Krylov–Bogoliubov–Mitropolsky method for nonlinear wave modulation

T. Kakutani and N. Sugimoto

Faculty of Engineering Science, Osaka University, Toyonaka, Osaka, Japan

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The Krylov–Bogoliubov–Mitropolsky perturbation method is applied to systems of nonlinear dispersive waves including plasma waves such as ion-acoustic, magneto-acoustic, and electron plasma waves. It is found that long time slow modulation of the complex wave amplitude can be described by the nonlinear Schrödinger equation for a very wide class of nonlinear dispersive waves.

I. INTRODUCTION

It is well known in the theory of nonlinear vibration that the Krylov–Bogoliubov–Mitropolsky perturbation method plays a powerful role in describing the long time behavior of the solution. The essential idea of the method consists of varying the amplitude so slowly that no secular terms can arise in the solution.

An extension of the Krylov–Bogoliubov–Mitropolsky method to systems of nonlinear partial differential equations was made by Montgomery and Tidman and also by Tidman and Stainer who calculated the amplitude-dependent wavenumber (or frequency) shifts for a monochromatic plane wave in plasmas. Similar results have also been obtained by Nayfeh by employing the derivative expansion method, which was devised by Sturrock and has been developed mainly in the field of statistical mechanics. In Refs. 2, 3, and 4, however, they made a simplifying (sufficient but not necessary) assumption which led them to fail to take into account the long time amplitude modulation. Improving Nayfeh's derivative expansion method, Kawahara has recently succeeded in taking into account not only the wavenumber (or frequency) shifts but also the long time slow modulation of the amplitude. He showed that the slow modulation of a monochromatic plane wave can be described by the so-called nonlinear Schrödinger equation which is familiar as the resultant equation of the reductive perturbation method developed by Taniguti and his collaborators. This type of equation has been obtained heuristically in the field of nonlinear optics and for a problem of the heat pulse in solids.

The main purpose of this article is to show that the Krylov–Bogoliubov–Mitropolsky method is also useful in obtaining the nonlinear Schrödinger equation for the amplitude modulation of a monochromatic plane wave as well as the reductive perturbation method or the derivative expansion method. An advantage of the Krylov–Bogoliubov–Mitropolsky method is that it is conceptually more natural than the derivative expansion method where one needs to introduce artificial multiplicity of the independent variables. The only assumption used in the Krylov–Bogoliubov–Mitropolsky method is just annihilation of secular terms and the method can suggest, quite naturally, a heuristic coordinate transformation on which the reductive perturbation method is based.

We first outline, in Sec. II, an extension of the Krylov–Bogoliubov–Mitropolsky method to a system of partial differential equations by using a simple model equation. It is found that a plane wave solution to the nonlinear Schrödinger equation gives a “nonlinear” dispersion relation, i.e., an effect of finite amplitude to the dispersion relation between wavenumber $k$ and frequency $\omega$, which was called “amplitude dispersion” by Lighthill. We then apply the method, in Sec. III, to actual physical systems such as ion-acoustic, magneto-acoustic, and electron plasma waves in collision-free plasmas. It is found that the ion-acoustic waves with short wavelength are modulationally unstable; more precisely, they are unstable for wavenumbers more than the critical value $k \cong 1.47/\lambda D$ where $\lambda D$ is the Debye length. On the other hand, the magneto-acoustic waves across a magnetic field and electron plasma waves are always stable for all wavelengths. Finally, some concluding remarks concerning competition between the effect of nonlinearity and that of dispersion are given in Sec. IV.

II. EXTENSION OF THE KRYLOV–BOGOLOUBOV–MITROPOLSKY METHOD

We shall first demonstrate how the Krylov–Bogoliubov–Mitropolsky method can be extended to a partial differential equation with a small nonlinear term. To do so, we deal with the following equation not only as a simple mathematical model but also as an approximate equation for water waves, lattice waves, and so forth:

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial t^2 \partial x^2} \right) f = \frac{\partial^2}{\partial t^2} \left( \frac{f^3}{2} \right),$$

(1)

where $f(x, t)$ is a real function of the one-dimensional space coordinate $x$ and the time $t$, and represents a small but finite perturbation from a uniform state. We choose a monochromatic plane wave solution $f_1$ to Eq. (1) as a starting solution setting the right-hand side equal to zero,

$$f_1 = a \exp(i\psi) + a^* \exp(-i\psi),$$

(2)

where $a$ is the complex amplitude, $\psi$ is the phase factor defined as $\psi = kx - \omega t$, and $k$ and $\omega$ being, respectively, the wavenumber and frequency, and $a^*$ denotes the complex conjugate to $a$. In order that $f_1$ not be trivial, the following linear dispersion relation should be satisfied:

$$D(k, \omega) = -\omega^2 + k^2 - \omega^2 k^2 = 0.$$  

(3)

We now seek the Krylov–Bogoliubov–Mitropolsky perturbation solution of the following form:

$$f = \epsilon f_1(a, \beta, \psi) + \epsilon^2 f_2(a, \beta, \psi) + \epsilon^3 f_3(a, \beta, \psi) + \cdots.$$  

(4)
In the above expression, \(a\), \(\dot{a}\), and \(\psi\) in the parentheses indicate that \(f_1, f_2, \ldots\) depend on \(x\) and \(t\) only through \(a\), \(\dot{a}\), and \(\psi\), where the complex amplitude \(d\) is assumed to be a slowly varying function of \(x\) and \(t\) through the relations

\[
\frac{\partial a}{\partial t} = \varepsilon A_1(a, \dot{a}) + \varepsilon^2 A_2(a, \dot{a}) + \cdots, \tag{5}
\]

\[
\frac{\partial a}{\partial x} = \varepsilon B_1(a, \dot{a}) + \varepsilon^2 B_2(a, \dot{a}) + \cdots, \tag{6}
\]

together with the complex conjugate relations to (5) and (6), where the phase \(\psi\) remains unchanged from the linearized limit, i.e., \(\psi = kx - \omega t\), because nonlinear effects on the phase may be taken into account through the “phase part” of the complex amplitude. The unknown functions \(A_1, B_1, A_2, B_2, \ldots\) should be determined so as to make the solution (4) free from secular terms.

Substituting (4) into the left-hand side (linear part) of Eq. (1) and collecting terms with the same power in \(\varepsilon\), we obtain

\[
\frac{\partial^2 \psi}{\partial \psi^2} \left( \frac{\partial^2}{\partial a\partial \dot{a}} - \frac{\partial^2}{\partial \dot{a}\partial \psi} \right) f \phi
= \varepsilon^2 \left[ i \left( \frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 \right) \exp(i\psi) + \text{c.c.} \right]
- \omega^2 \left( \frac{\partial f_2}{\partial \psi} + \frac{\partial f_1}{\partial \dot{a}} \right) + \varepsilon^4 \left[ i \left( \frac{\partial D}{\partial a} A_2 - \frac{\partial D}{\partial \dot{a}} B_2 \right) \right]
\times \exp(i\psi) - \frac{1}{2} \left[ \frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 \right] \left( A_1 \frac{\partial A_1}{\partial a} + A_1 \frac{\partial A_1}{\partial \dot{a}} \right)
- \frac{2}{\omega^2} \frac{\partial D}{\partial a} \left( \frac{\partial B_1}{\partial a} + \frac{\partial B_1}{\partial \dot{a}} \right) \exp(i\psi)
\times \exp(i\psi) + \frac{1}{\omega^2} \left[ \frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 \right] \left( A_1 \frac{\partial A_1}{\partial a} + A_1 \frac{\partial A_1}{\partial \dot{a}} \right)
- \frac{2}{\omega^2} \frac{\partial D}{\partial a} \left( \frac{\partial B_1}{\partial a} + \frac{\partial B_1}{\partial \dot{a}} \right) \exp(i\psi)
- \frac{2}{\omega^2} \frac{\partial D}{\partial a} \left( \frac{\partial B_1}{\partial a} + \frac{\partial B_1}{\partial \dot{a}} \right) \exp(i\psi)
- \omega^2 \left( \frac{\partial f_2}{\partial \dot{a}} \right) + \varepsilon^2 \left[ i \left( \frac{\partial D}{\partial a} A_2 - \frac{\partial D}{\partial \dot{a}} B_2 \right) \right]
\times \exp(i\psi) + \frac{1}{\omega^2} \left[ \frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 \right] \left( A_1 \frac{\partial A_1}{\partial a} + A_1 \frac{\partial A_1}{\partial \dot{a}} \right)
- \frac{2}{\omega^2} \frac{\partial D}{\partial a} \left( \frac{\partial B_1}{\partial a} + \frac{\partial B_1}{\partial \dot{a}} \right) \exp(i\psi)
- \frac{2}{\omega^2} \frac{\partial D}{\partial a} \left( \frac{\partial B_1}{\partial a} + \frac{\partial B_1}{\partial \dot{a}} \right) \exp(i\psi)
- \omega^2 \left( \frac{\partial f_2}{\partial \psi} + \frac{\partial f_1}{\partial \dot{a}} \right) + O(\varepsilon^2), \tag{7}
\]

where c.c. stands for the complex conjugate and the relations

\[
\frac{\partial D}{\partial a} = -2\omega(1 + k^2), \quad \frac{\partial D}{\partial \dot{a}} = 2k(1 - \omega^2),
\]

\[
\frac{\partial^2 D}{\partial \omega^2} = -2(1 + k^2), \quad \frac{\partial^2 D}{\partial \omega \partial k} = -4\omega k,
\]

\[
\frac{\partial^2 D}{\partial k^2} = 2(1 - \omega^2), \tag{8}
\]

have been used. On the other hand, substituting (4) into the right-hand side (nonlinear part) of Eq. (1), and using the relations (5) and (6), we can express the nonlinear term as the power series in \(\varepsilon\) as is the case for the linear part:

\[
\frac{\partial^2 \psi}{\partial \psi^2} \left( \frac{\partial^2}{\partial a\partial \dot{a}} - \frac{\partial^2}{\partial \dot{a}\partial \psi} \right) f \phi
= \varepsilon^2 \left[ k^2 \frac{\partial^2}{\partial \psi^2} \left( \frac{f_1^2}{2} \right) \right] + \varepsilon^4 \left[ 2kB_1 \frac{\partial}{\partial a} \frac{\partial}{\partial \psi} \left( \frac{f_1^2}{2} \right) \right] + \text{c.c.}
+ k^2 \frac{\partial}{\partial \psi} \left( f_1 f_2 \right) + O(\varepsilon^6). \tag{9}
\]

Let us first seek the second-order solution \(f_2\). Equating the coefficients of \(\varepsilon^2\) in (7) and (9), we have the following ordinary differential equation for \(f_2\) with respect to \(\psi\):

\[
k^2\omega^2 \left( \frac{\partial^2 f_2}{\partial \psi^2} + \frac{\partial f_2}{\partial \psi} \right) = i \left( \frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 \right) \exp(i\psi)
+ 2k^2a^2 \exp(2i\psi) + \text{c.c.}, \tag{10}
\]

where \(f_1\) in (9) has been replaced by its explicit form given by (2). The solution \(f_2\) would contain secular terms, i.e., \(\psi\) proportional terms unless the coefficients of \(\exp(i\psi)\) and \(\exp(-i\psi)\) in (10) vanish. If the condition for non-secularity

\[
\frac{\partial D}{\partial a} A_1 - \frac{\partial D}{\partial \dot{a}} B_1 = 0,
\]

or

\[
A_1 + V_\alpha B_1 = 0,
\]

\[
V_\alpha = -\frac{\partial D}{\partial k} \left/ \frac{\partial D}{\partial \omega} \right. = \frac{\omega^2}{k^2}, \tag{11}
\]

and its complex conjugate relations are satisfied, where \(V_\alpha\) represents the group velocity of the wave, we can obtain the secular-free solution \(f_2\) as follows:

\[
f_2 = \frac{1}{i\omega} \left[ a^2 \exp(2i\psi) + a^2 \exp(-2i\psi) \right] + b(a, \dot{a})
\times \exp(i\psi) + b(a, \dot{a}) \exp(-i\psi) + c(a, \dot{a}), \tag{12}
\]

where \(b(a, \dot{a}), \dot{b}(a, \dot{a})\) (assumed to be complex) and \(c(a, \dot{a})\) (assumed to be real) are constants with respect to \(\psi\) and \(\dot{\psi}\) should be determined as functions of \(a\) and \(\dot{a}\) from non-secular conditions in higher orders. Following Bogoliubov and Mitropolsky,\(^1\) we may set \(b(a, \dot{a}), \dot{b}(a, \dot{a})\) equal to zero, if we assume that the first harmonics, \(\exp(\pm i\psi)\), do not appear in the higher-order solutions. However, as will be shown later, it is not necessary to do so within the order of approximation considered in this paper.

Montgomery and Tidman\(^2\) and Tidman and Stainer\(^3\) concluded from (11) that \(A_1 = 0\) and \(B_1 = 0\). They are sufficient but not necessary conditions in order to avoid the secular terms. This is why they could not take account of the long time amplitude modulation. By virtue of the relations (5) and (6), it turns out that \(A_1\) and \(B_1\) may be regarded, respectively, as \(d \alpha / d t\) and \(\alpha / \partial a / \partial \dot{x}_1\) to the lowest order in \(\varepsilon\), where \(x_1 = \epsilon x\) and \(s_1 = \epsilon x\). Thus, Eq. (11) may be interpreted as

\[
\frac{\partial a}{\partial t} + V_\alpha \frac{\partial a}{\partial s_1} \cong 0, \tag{13}
\]
which is nothing but Eq. (28) in Ref. 9 or Eq. (3.17) in Ref. 4, in which Nayfeh concluded from his Eq. (3.17) that \( \partial a/\partial x_1 = 0 \) and \( \partial a/\partial x_2 = 0 \), which are sufficient but not necessary conditions in the same sense as discussed above. In fact, Eq. (13) shows that, to the lowest order in \( \epsilon \), the amplitude \( a \) is constant in a frame of reference moving with the group velocity, i.e., \( a \) depends on \( t \) and \( x \) only through \( \xi = x - V_\phi \), but neither \( \partial a/\partial t \), nor \( \partial a/\partial x_2 \) is necessarily zero separately.

Let us further proceed to the third-order solution \( f_3 \). As was carried out for the second-order solution \( f_2 \), equating the coefficients of \( \epsilon^4 \) in (7) and (9), we have an analogous equation for \( f_3 \) to Eq. (10) for \( f_2 \). For the solution \( f_3 \) to be secular-free, we must set the coefficients of \( \exp(\pm i \psi) \) in this equation equal to zero, giving rise to

\[
i (A_2 + V_\phi B_2) + \frac{1}{2} \frac{dV_\phi}{dk} \left( B_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial \bar{a}} \right) = -\frac{k^2}{2D/\partial \omega} \left( \frac{1}{6 \omega^2} \alpha^2 \bar{a}^2 + c \bar{a} \right),
\]

(14)

together with its complex conjugate relation, where the first-order relation (11), and the relation

\[
\frac{dV_\phi}{dk} = -\left( \frac{\partial \phi}{\partial \omega} + 2V_\phi \frac{\partial \phi}{\partial \omega} + V_\phi \frac{\partial \phi}{\partial \omega} \right) / \partial \omega = -3 \omega^2 / k^2,
\]

(15)

have been used.

It should be noted that Eq. (14) does not contain the arbitrary constants \( b \) and \( \bar{b} \) so that we need not determine them so far as Eq. (14) is concerned. However, it does contain \( c \). Therefore, in order to complete Eq. (14) for \( a \) we must determine the functional form of \( c(a, \bar{a}) \) with respect to \( a \) and \( \bar{a} \). This can be done as follows. The secular-producing terms that have so far appeared in obtaining \( f_2 \) and \( f_3 \) were only terms proportional to \( \exp(\pm i \psi) \), i.e., resonant terms. However, it should be noted that, in general, constant terms, if they appear, with respect to \( \psi \) (the functions of \( a \) and \( \bar{a} \) alone) are also secular-producing and that secular terms resulting from constant terms grow faster than those due to resonant terms, because the former are proportional to \( \psi^2 \). Therefore, in addition to resonant terms, we must also require the nonsecular condition for constant terms. For the present example, such a secular-producing constant first appears in the fourth power in \( \epsilon \) that is, for the nonlinear part

\[
C_N = \left( B_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial \bar{a}} \right) \bar{a} + \text{c.c.} + 2B_1 \bar{B}_1,
\]

(16)

while for the linear part

\[
C_L = A_1 \frac{\partial}{\partial \bar{x}_1} \left( A_1 \frac{\partial}{\partial \bar{x}_1} \right) + A_1 \frac{\partial}{\partial \bar{x}_2} \left( A_1 \frac{\partial}{\partial \bar{x}_2} \right) + B_1 \frac{\partial}{\partial a} \left( B_1 \frac{\partial}{\partial \bar{a}} \right)
\]

\[
\quad - B_1 \frac{\partial}{\partial \bar{a}} \left( B_1 \frac{\partial}{\partial a} \right) + \text{c.c.,}
\]

(17)

where \( C_N \) and \( C_L \) denote, respectively, constant terms contained in \( \partial^2 \phi / \partial x_1 \partial x_2 \) (f/2) and in \( \partial^2 \phi / \partial \bar{x}_1 \partial \bar{x}_2 - \partial^2 \phi / \partial \bar{x}_1 \partial x_2 - \partial^2 \phi / \partial x_1 \partial \bar{x}_2 \) in the fourth power in \( \epsilon \). Therefore, for \( f_3 \) to be secular-free, one must at least require

\[
(V_\phi^2 - 1) \left[ B_1 \frac{\partial B_1}{\partial a} \left( B_1 \frac{\partial}{\partial a} \right) + B_1 \frac{\partial B_1}{\partial \bar{a}} \left( B_1 \frac{\partial}{\partial \bar{a}} \right) \right] + \text{c.c.}
\]

\[
= \left( B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}} \right) \bar{a} + \text{c.c.} + 2B_1 \bar{B}_1,
\]

(18)

where Eq. (11) has been used. On the other hand, the nonsecular condition for \( f_3 \) arising from the resonant terms will yield an equation for \( A_2 \) and \( B_2 \) which is analogous to Eq. (14) for \( A_2 \) and \( B_2 \). However, we shall not go into such a "fine structure," because we are interested in the lowest significant order of approximation. Equation (18) can be satisfied if we choose the unknown constant \( c \) as follows (see the Appendix):

\[
c(a, \bar{a}) = \frac{1}{V_\phi^2 - 1} a \bar{a} + \beta,
\]

(19)

where it is assumed that \( V_\phi \neq 1 \) which may always be the case for dispersive waves for which the group velocity is different from the phase velocity. In (19), \( \beta \) is an absolute constant not only with respect to \( \psi \) but also to \( a \) and \( \bar{a} \).

Introducing (19) into Eq. (14) and noting that \( A_2, B_2, \) and \( B_1 (\partial B_1 / \partial a) + \bar{B}_1 (\partial B_1 / \partial \bar{a}) \) can be interpreted, respectively, as \( \partial a/\partial x_1 - A_2 / \epsilon, \partial a/\partial x_2 - B_1 / \epsilon, \) and \( \partial \bar{a}/\partial x_2 \), where \( t_1 = \epsilon \bar{t}, x_2 = \epsilon x_1, \) and \( x_1 = \epsilon x_1, \) we obtain

\[
i \left( \frac{\partial}{\partial a} + V_\phi \frac{\partial}{\partial \bar{a}} \right) + p \frac{\partial \bar{a}}{\partial x_2} = Q \mid a \mid \bar{a} + Ra,
\]

(20)

where

\[
P = \frac{1}{2} \frac{dV_\phi}{dk},
\]

\[
Q = -\frac{k^2}{2D/\partial \omega} \left( \frac{1}{6 \omega^2} + \frac{1}{V_\phi^2 - 1} \right),
\]

(21)

\[
R = -\frac{k^2}{2D/\partial \omega} \beta,
\]

which coincides with Eq. (30.1) in Ref. 9 except for the notation. This shows that, in the frame moving with the group velocity, variation of the amplitude \( a \) is determined by the nonlinear interaction \( Q \mid a \mid \bar{a} \), the linear interaction \( Ra \), and the dispersion term \( P \partial \bar{a} / \partial x_2 \). In the analysis by Montgomery et al., and by Nayfeh, the dispersion term \( P \partial \bar{a} / \partial x_2 \) has been missed. It is easily seen that Eq. (20) can be transformed into the nonlinear Schrödinger equation

\[
i \left( \frac{\partial}{\partial \bar{t}} + P \frac{\partial \bar{a}}{\partial \bar{x}_2} \right) = Q \mid a \mid \bar{a} + Ra,
\]

(22)

provided we introduce the coordinate transformation defined as

\[
\xi = \frac{1}{\epsilon} (x_2 - V_\phi \bar{t}), \quad x_2 - V_\phi \bar{t} = \epsilon (x - V_\phi \bar{t}),
\]

\[
\tau = t_2 = \epsilon \bar{t},
\]

(23)
which is nothing but the heuristic coordinate transformation introduced by Taniuti et al.\cite{3,4,5,6} in their reductive perturbation method.

It should be noted that the arbitrary constant \( \beta \) (and therefore \( R \)) may be determined if appropriate initial and/or boundary conditions are specified. For example, if we impose the boundary condition that \( c(a, \beta) \) tends to zero as \( \xi \to -\infty \), which means that there is induced no steady part at infinity, then \( \beta \) is determined as \( \beta = -\alpha d_a/(V_{p}^2 - 1) \), \( \alpha \) being the value of \( a \) at \( \xi = -\infty \). It turns out, however, that the linear interaction term \( Ra \) is not so essential because it causes only a simple phase shift. In fact, the \( Ka \) term in Eq. (22) can be removed by a simple substitution\cite{20}

\[ a \to a \exp(-iRr). \]  

(24)

It is interesting to note that a plane wave solution to Eq. (22),

\[ a = A_0 \exp[i(K\xi - \Omega t)], \]  

(25)

where \( A_0 \) is a complex constant and \( K, \Omega \) are real constants, gives a dispersion relation

\[ \Omega - PK^2 = Q \, |A_0|^2 + R. \]  

(26)

On the other hand, since

\[ a \exp(i\psi) = A_0 \exp[i(\xi k + eK)x] \]

\[ - (\omega + eKV_a + e\Omega i)], \]  

(27)

we have

\[ \omega(k + \Delta k, A_0) \cong \omega(k + \Delta k) + e'(Q \, |A_0|^2 + R), \]  

(28)

where \( \omega(k + \Delta k) = \omega + V_a \Delta k + \frac{1}{2}(dV_a/dk)(\Delta k)^2 \) and \( \Delta k = \kappa K \). The second term of Eq. (28) shows that the effect of finite amplitude on the dispersion relation between \( k \) and \( \omega \), and corresponds to “amplitude dispersion” so called by Lighthill.\cite{9} In this sense, we may interpret Eq. (28) as a generalized dispersion relation for weakly nonlinear dispersive waves.

III. PLASMA WAVES

Our immediate motive in extending the Krylov-Bogoliubov-Mitropolsky method to systems of partial differential equations is to apply the method to various waves in collision-free plasmas which are typical dispersive media. Amongst them, we consider, in this paper, three different waves, i.e., ion-acoustic, magneto-acoustic, and electron plasma waves.

A. Ion-acoustic waves

Suppose we have one-dimensional ion-acoustic waves traveling in a collision-free plasma consisting of cold ions \((T_i = 0)\), and isothermal electrons \((T_e = \text{const} \neq 0)\). Neglecting the effects of Landau damping and electron inertia (note that the mass ratio \( m_e/m_i \ll 1 \) for all practical plasmas), we may describe the behavior of such a plasma by the two-fluid model, and the basic system of equations may be written in nondimensional form as\cite{23}

\[ \frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = E, \]

\[ \frac{\partial n_a}{\partial x} = -n_a E, \quad \frac{\partial E}{\partial x} = n - n_a, \]  

(29)

where \( n, n_a, u, \) and \( E \) denote, respectively, ion density, electron density, ion-fluid velocity, and the electric field, which are all normalized with respect to the undisturbed density \( N_0 \), the characteristic sound speed \((kT_e/m_e)^{1/2}\), \( \kappa \) being the Boltzmann constant, and the characteristic electric field \( kT_e/(e\lambda_d) \), where \( \lambda_d \) is the Debye length defined by \( \lambda_d = [kT_e/(4\pi e^2 N_0)]^{1/2} \), \( e \) being the unit electronic charge, whereby the characteristic length is chosen to be the Debye length.

Let us now expand all quantities around the unperturbed uniform state in the following form:

\[ \begin{bmatrix} E' \\ u \\ n \\ n_a \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 + \epsilon \\ 1 + \epsilon \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} + \cdots, \]  

(30)

and assume that each coefficient of the \( \epsilon \) power depends on \( x \) and \( t \) through \( a, \beta, \psi \) in the same sense as in Sec. II. Substituting expression (30) into the system of equations (29) and equating terms with the same power of \( \epsilon \), we arrive at the set of equations for each power in \( \epsilon \). From the set of equations for \( O(\epsilon) \), we may choose the starting solutions as

\[ E_1 = a \exp(i\psi) + \delta \exp(-i\psi), \]

\[ u_1 = \frac{i}{\omega} [a \exp(i\psi) - \delta \exp(-i\psi)], \]

\[ n_1 = \frac{i}{\omega} [a \exp(i\psi) - \delta \exp(-i\psi)], \]

\[ n_{a1} = \frac{i}{\kappa} [a \exp(i\psi) - \delta \exp(-i\psi)]. \]  

(31)

In order that the starting solutions be nontrivial, the dispersion relation between \( k \) and \( \omega \), which takes the same form for this example as that given by (3) should be satisfied.

The second set of equations for \( O(\epsilon^2) \) then gives the secular-free condition

\[ A_1 + V_p B_1 = 0, \]  

(32)

which corresponds to Eq. (11). Under this condition, we
have the second-order solutions as,

\[
E_2 = \frac{i(3k^4 - \omega^4)}{3\omega^3 k^3} a^2 \exp(2i\psi) + b \exp(i\psi) + \text{c.c.},
\]

\[
w_2 = -\frac{[3(\omega^4 + 1)k^4 - \omega^4]}{6\omega^3 k^3} a^2 \exp(2i\psi)
- \left(\frac{\omega}{k^3} B_1 - \frac{i}{\omega} b\right) \exp(i\psi) + \text{c.c.} + c_1,
\]

\[
w_{2a} = -\frac{[3(\omega^4 + 1)k^4 - \omega^4]}{6\omega^3 k^4} a^2 \exp(2i\psi)
+ \left(\frac{k^3 - 1}{k^3} B_1 + \frac{ik}{\omega} b\right) \exp(i\psi) + \text{c.c.} + c_3,
\]

\[
w_{2b} = -\frac{1}{k^3} B_1 - \frac{i}{k} b\right) \exp(i\psi) + \text{c.c.} + c_3, \quad (33)
\]

where \(b, b\) (assumed to be complex) and \(c_1, c_2, c_3\) and \(c_4\) (assumed to be real) are arbitrary constants with respect to \(\psi\) and depend on \(a\) and \(b\) alone. They should be determined from the nonsecular conditions at higher orders. It is found, from the second set of equations for \(O(\epsilon^4)\), that \(E_2\) has no constant term and that the constant \(c_2\) contained in \(n_2\) should be equal to the constant \(c_2\) contained in \(w_{2b}\), so that

\[
c_2(a, \bar{a}) = c_2(a, \bar{a}). \quad (34)
\]

It can be seen from the third set of equations for \(O(\epsilon^4)\) that the secular-free conditions for the third-order solutions consist of two parts: One is the annihilation of the secular-producing constants and the other is that of secular-producing resonant terms. From the former, we can determine \(c_1, c_2\) as follows:

\[
c_1 = \frac{(2k/\omega^2) + V_0}{V_0^2 - 1} \frac{\alpha\bar{a}}{} + \beta_1,
\]

\[
c_2 = \frac{(2k/\omega^2) V_0 + 1}{V_0^2 - 1} \frac{\alpha\bar{a}}{} + \beta_2, \quad (35)
\]

where \(\beta_1, \beta_2\) are arbitrary absolute constants corresponding to \(\beta\) in the preceding section [cf. (19)]. On the other hand, elimination of resonant terms gives rise to

\[
i\left(A_2 + \frac{\omega}{k^3} B_2\right) - \frac{3\omega\omega}{2k^4} B_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial \bar{a}}
= \frac{\omega^2(3k^{10} + 6k^6 - 6k^6 - 29k^4 - 30k^4 - 12)}{12k^4(k^4 + 3k^2 + 3)} a^2 d
\]

\[
+ \left(\frac{k^3}{2} \beta_1 \right) a, \quad (36)
\]

which can be transformed into the nonlinear Schrödinger equation of the form of Eq. (22) with the coefficients

\[
P = \frac{1}{2} \frac{dV_s}{dk} = -\frac{3\omega^5}{2k^4}, \quad (37)
\]

\[
Q = \omega^3 \left(3k^{10} + 6k^6 - 6k^6 - 29k^4 - 30k^4 - 12\right),
\]

\[
R = k^3 \left(\beta_1 + \frac{\omega^2}{2} \beta_2, \quad (33)
\]

which should be compared with those obtained by the reductive perturbation method.22,26 In Ref. 22, Shimizu and Ichikawa obtained the nonlinear Schrödinger equation for the amplitude of ion density based on the fluid model, while, in Ref. 23, Ichikawa et al. obtained the nonlinear Schrödinger equation for the amplitude of electric potential based on the Vlasov model. Except for this trivial difference, the present result coincides exactly with that of Ref. 22 and with that of Ref. 23 in the limit of \(T_i \to 0\).

As shown by Taniuti and Yajima27 and also by Hasimoto and Ono,28 the plane wave solution of the nonlinear Schrödinger equation is modulationally unstable if \(PQ < 0\). Inspection of (37) shows that \(PQ \equiv 0\) according as \(k \leq k_c \equiv 1.47\) so that short waves with \(k > k_c\) are modulationally unstable, while long waves with \(k < k_c\) are stable as expected. In fact, the ion-acoustic waves with long wave length can be described approximately by Eq. (1) considered in Sec. II, and \(PQ\) is found to always be positive for \(P\) and \(Q\) given by (21).

B. Magneto-acoustic waves

As the second example, we consider the magneto-acoustic waves propagating across a magnetic field in a cold collision-free plasma. Such a plasma may be described by a framework of “magneto-ion dynamics,”29 in which the field quantities are reduced to ion density \(n\), ion-fluid velocity \(v\), and the magnetic field \(B\) alone; i.e., all other quantities such as electron density, electron-fluid velocity, and electric field can be eliminated. Let us restrict ourselves to one-dimensional plane waves propagating along the \(x\) direction, while the applied magnetic field is chosen so as to lie along the \(x\) direction. Since we consider the waves traveling across the magnetic field, we may set \(\mathbf{v} = \mathbf{v}(u, v, 0)\) and \(\mathbf{B} = \mathbf{B}(0, 0, B_0)\) so that the basic system of equations may be written in nondimensional form as

\[
\frac{\partial u}{\partial t} + \frac{\partial (nu)}{\partial x} = 0,
\]

\[
\frac{\partial u}{\partial t} + \frac{1}{n} \frac{\partial (B_x^2)}{\partial x} = 0, \quad \frac{\partial v}{\partial t} = -\frac{1}{R_e} \frac{\partial}{\partial x} \left(\frac{1}{n} \frac{\partial B_x}{\partial x} \right), \quad (38)
\]

\[
\frac{\partial B_x}{\partial t} + B_x \frac{\partial u}{\partial x} = -\frac{1}{R_s} \frac{\partial v}{\partial x} - \frac{\partial \alpha}{\partial x},
\]

where \(n, v, B\) are normalized, respectively, by the undisturbed density \(N_0\), Alfvén speed \(B_0/[4\pi (n_0 + m_0)] V_0\), and the strength of the applied magnetic field \(B_0, m, n,\) being, respectively, the masses of the ions and of the electrons. The nondimensional parameters \(R_s, R_e, \) denote, respectively, the normalized ion and electron Larmor frequencies, where the characteristic frequency has been
chosen as the geometric mean of ion and electron Larmor frequencies, \( eB_0/\sqrt{(e/m_i)^{1/2}} \), so that \( R_e = (m_i/m_e)^{1/2} \) and \( R_s = (m_i/m_e)^{1/2}, \) \( \epsilon \) and \( c \) being, respectively, the unit electronic charge and the speed of light.

As was carried out in Sec. IIIA, we expand all quantities as a power series in \( \epsilon, \)

\[
\begin{bmatrix}
B_z \\
B_{z1} \\
B_{z2} \\
B_{z3} \\
B_e \\
B_{el}
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 + \epsilon \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
u_1 \\
u_2 \\
\epsilon
\end{bmatrix} +
\begin{bmatrix}
e^0 \\
e^1 \\
e^2 \\
\epsilon
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
+ \cdots,
\]

(39)

and rearrange the basic system of equations (38) in terms of powers in \( \epsilon. \) From the first set of equations for \( O(\epsilon), \) we have the starting solutions as follows:

\[
B_{z\epsilon} = a \exp(i\psi) + \epsilon \exp(-i\psi),
\]

\[
n_1 = \frac{k}{\omega} \left[ a \exp(i\psi) + \epsilon \exp(-i\psi) \right],
\]

\[
u_1 = \frac{k}{\omega} \left[ a \exp(i\psi) + \epsilon \exp(-i\psi) \right],
\]

\[
t_1 = -\frac{ik}{R_e} \left[ a \exp(i\psi) - \epsilon \exp(-i\psi) \right],
\]

(40)

where the dispersion relation again takes the same form as that given by (3). It is found that the nonsecular condition for the second-order solutions coincides with that given by (11) or (32). Thus, the secular-free second-order solutions can be obtained as

\[
B_{z\epsilon} = \frac{2k^2 + 1}{2\omega^2} a^2 \exp(2i\psi) + \epsilon a \exp(i\psi) + c.c. + c_1,
\]

\[
n_2 = \frac{k(4k^2 + \omega^2 + 1)}{2\omega^3} a^2 \exp(2i\psi)
- \left( \frac{2ikB_1 - k^2}{\omega^2} \right) a \exp(i\psi) + c.c. + c_2,
\]

\[
u_2 = \frac{k(2k^2 + \omega^2 + 1)}{2\omega^3} a^2 \exp(2i\psi)
- \left( \frac{i\omega B_1 - k}{\omega} \right) a \exp(i\psi) + c.c. + c_3,
\]

\[
t_2 = \frac{-ik(k^2 + 1)}{R_e\omega^2} a^2 \exp(2i\psi)
- \frac{1}{R_e} \left( B_1 + i\omega b \right) a \exp(i\psi) + c.c. + c_4
\]

(41)

where \( b, \ b \) (assumed to be complex) and \( c_1, c_2, c_3, \) and \( c_4 \) (assumed to be real) are arbitrary constants with respect to \( \psi. \) From the nonsecular condition for the third-order solutions, we can determine \( c_1, c_2, c_3, \) and \( c_4 \) as

\[
c_1 = \frac{2k^2 + 3}{V_e^2 - 1} a \delta + \beta_1,
\]

\[
c_2 = \frac{2k^2 + 3}{V_e^2 - 1} \epsilon a \delta + \beta_2,
\]

\[
c_3 = \frac{(2k^2 + 4)V_e^2 + 2}{V_e(V_e^2 - 1)} \epsilon a \delta + \beta_3,
\]

\[
c_4 = \beta_4,
\]

(42)

where \( \beta_1, \beta_2, \beta_3, \) and \( \beta_4 \) are arbitrary absolute constants. The nonsecular condition with respect to resonant terms gives

\[
i \left( A_2 + \frac{\omega^3}{k^2} B_2 \right) - \frac{3\omega^3}{2k^2} \left( B_1 \delta \frac{\partial B_1}{\partial a} + \beta_1 \frac{\partial B_1}{\partial \beta_1} \right)
- \frac{4\omega(k^2 + 1)(k^3 + 3k^2 + 3)}{4\omega(k^2 + 1)} a \delta
- \left( \omega \beta_1 - \frac{\omega^3}{2k^2} \beta_2 + \delta \beta_3 \right)
\]

(43)

which can be reduced to the form of the nonlinear Schrödinger equation (22) with the coefficients given by

\[
P = \frac{1}{2} \frac{dV_e}{dk} = -\frac{3\omega^3}{2k^4},
\]

\[
Q = -\frac{(4k^4 + 11k^2 + 15k^2 + 9)}{4\omega(k^2 + 1)(k^3 + 3k^2 + 3)},
\]

\[
R = \omega \beta_1 - \omega \beta_2(k^2 + 3k^2 + 3)
\]

(44)

which coincides exactly with the result obtained by the reductive perturbation method. Since \( PQ \) is always positive for this example, the magneto-acoustic waves across a magnetic field are modulationaly stable for all wave-numbers.

C. Electron plasma waves

As the last example, we consider the one-dimensional electron plasma waves in a warm electron plasma where ions are assumed to be the immobile positive background. There may be two different fluid models, one being the isothermal electron model which was adopted by Asano et al. and the other being the adiabatic electron model which is nothing but the problem treated by Tidman and Stainer and also by Nayfeh. Since one of the main reasons for reconsidering the problem is to comment on the analysis due to Tidman and Stainer and to Nayfeh, here, we shall employ the adiabatic model. Then, the basic system of equations may be written in nondimensional form as:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{3n} \frac{\partial p}{\partial x} + E = 0,
\]

\[
\frac{\partial p}{\partial t} + 3p \frac{\partial u}{\partial x} + u \frac{\partial p}{\partial x} = 0,
\]

\[
\frac{\partial E}{\partial t} = -nu = 0, \quad \frac{\partial E}{\partial x} = n - 1 = 0,
\]

(45)

where the usual continuity equation has been discarded, since it can readily be deduced from the last two equations in Eqs. (45). In the above equations, \( n, u, p, \) and \( E \) are electron number density, electron-fluid velocity, electron
pressure, and the electric field, respectively. They are normalized by the undisturbed density \(N_0\) (which is equal to the background ion density), the characteristic sound speed \(\sqrt{3p_0/(m_eeV_0)}\), the undisturbed pressure \(p_0\), and the characteristic electric field \((12\pi p_0)^{1/2}\), whereby the characteristic frequency is chosen as the frequency of electron plasma oscillation \(\omega_0 = (4\pi e^2 N_0/m_e)^{1/2}\), \(m_e\) and \(e\) being, respectively, electron mass and the unit electronic charge.

As before, we shall expand all quantities in terms of \(\epsilon\):

\[
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4
\end{bmatrix} = \begin{bmatrix}
0 \\
\epsilon n_1 \\
\epsilon^2 n_2 \\
\epsilon^3 n_3
\end{bmatrix} + \cdots
\]

and rearrange the basic system of Eqs. (45) in the order of powers of \(\epsilon\). We then choose the starting solutions as

\[
\begin{align*}
E_1 &= a \exp(i\psi) + b \exp(-i\psi), \\
\nu_1 &= -\omega a \exp(i\psi) - b \exp(-i\psi), \\
\nu_2 &= -i k a \exp(i\psi) - b \exp(-i\psi), \\
p_1 &= -i k a \exp(i\psi) - b \exp(-i\psi),
\end{align*}
\]

where the dispersion relation takes the form

\[
D(k, \omega) = -\omega^2 + k^2 + 1 = 0, \tag{48}
\]

which is different from that of Eq. (3). In spite of this difference, the nonsecular condition for the second-order problem again takes the same form as that given by (11) or (32), with \(V_0 = k/\omega\). Thus, the secular-free second-order solutions can be obtained as

\[
\begin{align*}
E_2 &= -\frac{i k (4k^2 + 3)}{3} a^2 \exp(2i\psi) + b \exp(i\psi) + c.c., \\
\nu_2 &= -\frac{\omega k (8k^2 + 3)}{3} a^2 \exp(2i\psi) \\
&\quad - \frac{k}{\omega} B_1 + i b \exp(i\psi) + c.c. - 2 \omega k a a, \\
n_2 &= -\frac{2k^2 (4k^2 + 3)}{3} a^2 \exp(2i\psi) \\
&\quad - (B_1 + i k b) \exp(i\psi) + c.c., \\
p_2 &= -k^2 (8k^2 + 9) a^2 \exp(2i\psi) \\
&\quad - 3 (B_1 + i k b) \exp(i\psi) + c.c. - 6 k^2 a a + c,
\end{align*}
\]

where arbitrary constants \(b, \bar{b}\) (assumed to be complex) and \(c\) (assumed to be real) are determined as functions of \(a\) and \(\bar{a}\) from the higher order nonsecular conditions. It is found, from the second set of equations for \(O(\epsilon^2)\), that no arbitrary constant in \(E_2, \nu_2\) and \(n_2\) arises, but \(p_2\) contains the arbitrary constant \(\tilde{c}\) which was chosen by Tidman and Stainer, and also by Nayfeh as \(\tilde{c} = 6k^2 a a\) for some reason not explained. We shall use this freedom to annihilate secular-producing constant terms in the third order in \(\epsilon\). In fact, we can determine \(\tilde{c}\) as

\[
\tilde{c} = 12k^2 a a + \beta, \tag{50}
\]

where \(\beta\) is an arbitrary absolute constant. The nonsecular condition for the resonant terms then gives

\[
\begin{align*}
&\frac{1}{2} i \left( A_1 - \frac{k}{\omega} B_1 \right) + \frac{1}{2} \frac{1}{\omega} \left( B_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial a^2} \right) \\
&= \frac{k^2 (16k^2 + 15)}{3 \omega} a^2 a + \frac{k^2}{2 \omega} \beta a,
\end{align*}
\]

which can be reduced to the form of the nonlinear Schrödinger equation (22) with the coefficients given by

\[
P = \frac{1}{2} \frac{dV_0}{da} = \frac{1}{2} \frac{k^2 (16k^2 + 15)}{3 \omega}, \quad Q = \frac{k^2}{2 \omega} \beta, \quad R = \frac{k^2}{2 \omega} \beta,
\]

which should be compared with the results obtained by Tidman and Stainer and by Nayfeh. It can be seen that in these papers not only the dispersion term has been missed, which corresponds to \(P = 0\), but also the expression for \(Q\) is different from that given in (52), the former being due to the simplifying assumption mentioned earlier and the latter due to the difference in the value of \(c\).

Since \(PQ\) is always positive for any value of \(k\), the electron plasma waves are modulational stable for all wavelengths.

**IV. CONCLUDING REMARKS**

We have shown that the Krylov–Bogoliubov–Mitropolsky method, which was originally devised for systems of ordinary differential equations, can be effectively extended to systems of partial differential equations governing weakly nonlinear dispersive waves. Applying this method to several systems for obtaining nonlinear wave modulation, we have found that, to the lowest order of approximation, the complex amplitude of waves is constant in a frame of reference moving with the group velocity and that, to the next order of approximation, it is described by the nonlinear Schrödinger equation in the above frame of reference. The results thus obtained are in good agreement with those obtained by the reductive perturbation method and by the derivative expansion method. Thus, these three different methods are found to be equivalent to each other so far as the resultant equation is concerned. Although in this paper we have dealt with some particular examples, we may easily extend the present method to other nonlinear dispersive systems such as water waves, lattice waves, and so on.

It should be remarked here that the present analysis is concerned with the long time behavior of a single "quasimonochromatic" plane wave; that is, we are concerned with the asymptotic behavior of a forward progressing narrow-bandwidth wave packet. However, the present method may also be extended to weakly nonlinear systems in which multiwave interactions such as the harmonic or the parametric resonances do occur. For example, for harmonic resonant systems, one must replace the start-
ing solution by the superposition of the fundamental mode and its nth harmonic (n being an integer satisfying n ≥ 2) instead of a single monochromatic wave, where (k, ω) satisfies the resonant condition D(ak, nω) = 0 in addition to the linear dispersion relation D(k, ω) = 0. For parametric-resonant systems, one must replace the starting solution by the superposition of the three waves with (k_j, ω_j), j = 1, 2, and 3, for which the resonant conditions k_1 + k_2 = k_3 and ω_1 + ω_2 = ω_3 hold simultaneously in addition to the linear dispersion relation D(k_j, ω_j) = 0. All these problems are concerned with interactions among a small number of specific narrow bandwidth wavepackets centered around particular values of (k, ω). On the other hand, if one attempts to deal with general initial and/or boundary value problems, one must consider a complete set of monochromatic waves instead of a single wave or a small number of specific waves. Such an attempt has been made by Montgomery for a system described by the nonlinear Klein–Gordon equation.

At this stage, let us consider competition between the effect of nonlinearity and that of dispersion. Defining

$$a = p^{1/2} \exp \left( \frac{i}{2P} \int \sigma \, d\xi \right),$$

\(\rho(\xi, \tau)\) and \(\sigma(\xi, \tau)\) being real functions of \(\xi\) and \(\tau\), we can rewrite the nonlinear Schrödinger equation (22) in the following real form:

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial \xi} \left( p \sigma \right) = 0,$$

where the unimportant linear interaction term has been discarded, i.e., for simplicity, \(R = 0\) has been set equal to zero. As can easily be seen from this set of equations, if \(P = 0\), the solution will generally steepen the waveform in the course of time evolution and eventually break down, so that the equation obtained in Refs. 2, 3, and 4 is not adequate to describe the long time behavior of the wave modulation. On the other hand, if \(P \neq 0\), the “dispersion term”

$$p^{1/2} \frac{\partial}{\partial \xi} \left[ \rho^{-1/2} \frac{\partial}{\partial \xi} \left( \rho^{-1/2} \frac{\partial \rho}{\partial \xi} \right) \right]$$

begins to play a role in checking the steepening of the waveform, so that one may expect some equilibrium state resulting from the balance between nonlinearity and dispersion. In fact, as was shown by Asano et al. and also by Hasimoto and Ono, Eq. (22) has the following type of “equilibrium” solution (again setting \(R = 0\)):

$$A(\xi) = A(\xi) \exp(i\alpha \xi),$$

where \(A(\xi)\) is a real function of \(\xi\), and \(\alpha\) is a real constant, provided that \(A(\xi)\) satisfies the following equation:

$$\frac{1}{2} \left( \frac{dA}{d\xi} \right)^2 + U(A) = E_0, \quad U(A) = -\frac{Q}{4P} A^4 - \frac{\alpha}{2P} A^2,$$

where \(E_0\) is an arbitrary constant. This is equivalent to the classical equation of motion for a unit mass with total energy \(E_0\) under the potential \(U(A)\). By virtue of this analogy, it is easily found that if \(PQ > 0\) and \(\alpha P < 0\) (and therefore \(\alpha Q < 0\)), \(A(\xi)\) represents wavetrains expressible by the Jacobian elliptic functions (the explicit functional form is not so interesting to write down), provided that \(0 < E_0 < \alpha^2/(4PQ)\). If, in particular, \(E_0 = \alpha^2/(4PQ)\), then \(A(\xi)\) represents the “phase jump” expressed as

$$A(\xi) = \left( -\frac{\alpha}{Q} \right)^{1/2} \tanh \left( -\frac{\alpha}{2P} \right)^{1/2} \xi. \quad (56)$$

There is no bounded real solution for \(PQ > 0\) and \(\alpha P > 0\). On the other hand, if \(PQ < 0\) and \(\alpha P > 0\) (and therefore \(\alpha Q > 0\)), \(A(\xi)\) represents two types of wavetrains: One is the “large” amplitude waves for \(E_0 > 0\) and the other the “small” amplitude waves for \(\alpha^2/(4PQ) < E_0 < 0\), both may be expressed in terms of the Jacobian elliptic functions. If, in particular, \(E_0 = 0\), then \(A(\xi)\) represents a solitary wave expressed as

$$A(\xi) = \left( -\frac{2\alpha}{Q} \right)^{1/2} \text{sech} \left( \frac{\alpha}{P} \right)^{1/2} \xi. \quad (57)$$

The only possible solutions for the last case \(PQ < 0\) and \(\alpha P < 0\) are wavetrains again expressible by the Jacobian elliptic functions for \(E_0 > 0\).

In concluding this section, it should be mentioned that for \(PQ < 0\), an interesting initial value problem has been solved numerically by Yajima and Outi, who showed that the solitary wave solution such as given by (57) is so stable that it preserves its identity in spite of the nonlinear interactions. On account of this, the solitary wave is often called the “envelope soliton,” which should be compared with the Korteweg–de Vries soliton. A similar initial value problem has also been solved analytically by Zakharov and Shabat by using a technique of the inverse scattering problem, and they showed that a fairly large class of initial disturbances develops into a sequence of envelope solitons.

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APPENDIX: REMARKS ON THE CONSTANT \(\beta\)

We have assumed that the constant \(\beta\) in (19) is independent of both \(y\) and \(a\) in the main text. It seems necessary, however, to add some remarks on the constancy of \(\beta\). In view of the expression (6) and its complex conjugate relation, Eq. (18) may be interpreted as

$$\frac{\partial^2 c}{\partial s^2} = \frac{1}{V_s^2 - 1} \frac{\partial^2}{\partial s^2} (\alpha d),$$

(1A)
where $x_t = \epsilon x$. This can immediately be integrated to give

$$
\frac{c}{V_\epsilon} = \frac{1}{\alpha x_t + \beta},
$$

(A2)

where $\alpha$ and $\beta$ are integration constants independent of $x_t$. By virtue of (11), we can replace $B_1$ in Eq. (18) by $A_1$ and then Eq. (A2) may be replaced by

$$
\frac{c}{V_\epsilon} = \frac{1}{\alpha x_t + \alpha t_1 + \beta},
$$

(A3)

where the constants $\alpha$ and $\beta$ are independent of $t_1 (= \epsilon t)$. It should be noted, however, that the constant $\alpha$ (or $\alpha t_1$) should be set equal to zero, because the term $\alpha x_t$ (or $\alpha t_1$) is secular with respect to $x_t$ (or $t_1$). Combining Eqs. (A2) and (A3) after setting $\alpha = \alpha t_1 = 0$, we have finally

$$
\frac{c}{V_\epsilon} = \frac{1}{\alpha x_t + \beta},
$$

(A4)

where $\beta$ is independent of both $x_t$ and $t_1$, and therefore independent of $x_t (= x_t - V_\epsilon t_1)$.

Although $\beta$ is independent of $x_t$ and $t_1$, it may generally depend on $x_t (= \epsilon x)$, $t_1 (= \epsilon t)$, $x_t (= \epsilon x)$, $t_1 (= \epsilon t)$, $\cdots$, in the sense of multiple scale. On the other hand, the independent variables of the final nonlinear Schrödinger equation (22) are $x_t (= x_t - V_\epsilon t_1)$ and $\tau (= t_1)$. Therefore, if $\beta$ depends on $x_t$ and $t_1$, we need some modification to interpret the final Eq. (22). The dependence of $\beta$ on $x_t$ may be interpreted as parametric, more precisely, since $x_t = V_\epsilon t + \epsilon x$ by (23), the difference between $x_t$ and $V_\epsilon t$ is higher order so far as one regards $\epsilon$ as $O(1)$, i.e., $\epsilon x$ is the same order as $x_t$ and $t_1$, while the dependence of $\beta$ on $t_1$ (or $\tau$) seems crucial. It turns out, however, that the dependence of $\beta$ (as that of $R$) on $\tau$ may easily be removed by a simple substitution

$$
a \rightarrow a \exp \left[-\int R(\tau') \, d\tau'\right]
$$

(A5)

once the “constant” $\beta$ is determined from appropriate initial and/or boundary conditions [cf. (24)]. The same discussion can, of course, be applied to $\beta$'s in (35), (42), and (50).