Nonlinear Modulation of Torsional Waves in Elastic Rod

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Nonlinear Schrödinger equation, which describes the nonlinear modulation of dispersive torsional waves in an elastic rod of circular cross-section, is derived by the derivative expansion method. It is found, for the lowest dispersive mode, that the modulational instability occurs except in the range of the carrier wavenumber, \(2.799 < k < 3.032\). However, at the critical wavenumber where the fundamental and its second-harmonic waves can propagate simultaneously, the second-harmonic resonance takes place and then the nonlinear Schrödinger equation is no longer valid. In this case, another system of equations is derived, which governs both the wave amplitudes involved in this resonance between the fundamental torsional and its second-harmonic longitudinal modes.

§1. Introduction

The far-field theory of weakly nonlinear waves has been developed considerably\(^1\) in the field of fluid dynamics and plasma physics, in which a balance between the nonlinearity and the dispersion (or dissipation) does hold. As is well known, the evolution of the envelope of quasi-monochromatic carrier waves, which is roughly travelling with the group velocity, is governed by the nonlinear Schrödinger equation as the result of balance between the nonlinearity and the dispersion, where the wave amplitude and the sideband width of the carrier wavenumber have the same order of magnitude. This phenomenon, referred to as the self-modulation, is common to most of dispersive wave systems such as water waves\(^2\) and plasma waves.\(^3\) The problem of nonlinear modulational instability may be discussed on the basis of this equation. There arise, however, some exceptional cases where the usual perturbation scheme fails to derive the nonlinear Schrödinger equation. They are the so-called harmonic resonances,\(^*\) which result from the resonance between the fundamental and its higher harmonic modes whose phase velocities coincide with each other at a critical wavenumber. The second-harmonic resonance, which is the simplest case of such resonances, is known to take place for the capillary-gravity waves on deep water\(^6\) and the hydromagnetic waves in a cold collisionless plasma.\(^7\) The slowly varying amplitudes of the two waves in resonance are then described by a set of dynamical equations coupled through the nonlinear terms.

In the field of solid mechanics, however, the far-field theory has hardly been applied even to elasticity except for Nariboli's pioneering work.\(^8\) The main purpose of this paper is to extend this sort of asymptotic theory to the nonlinear wave modulation and its associated instability for dispersive elastic waves. As is well known from the classical linear elasticity, an unbounded homogeneous isotropic elastic body can support only two nondispersive wave modes, compression and shear. Therefore the dispersion of elastic waves should be attributed to the presence of boundary at which repeated reflexions take place. Thus the dispersive character in elastic waves depends upon the geometrical configurations such as a sphere, a circular cylinder and so on.\(^9\) To set about our investigation, we shall be concerned mainly with the dispersive torsional mode in the circular rod of homogeneous isotropic elastic media, because it is the simplest example which represents the essence of dispersive elastic waves.

On the other hand, the basic formulation for the nonlinear elasticity of general continuum has been made by many authors (see Truesdell and Noll,\(^10\) for example) but its application to specific problems has not been made so much. When small but finite deforma-
tions are taken into account, the nonlinearity appears in the strain tensor and the constitutive equation. Moreover it is necessary to distinguish between the Kirchhoff’s and the Lagrangian stress tensors. In the constitutive equation expanded with respect to the strain, the third-order elasticity will be here introduced in addition to the linear elasticity (the second-order elasticity). Taking the third-order elasticity into consideration, Huki and Mukai\cite{11} have treated the interaction of intersecting elastic waves in Cu single crystals from a quantum mechanical point of view. It is explicated there that such an interaction originates from the nonlinearity of the lattice vibrations which is measured by the third-order elasticity. It will be seen also that this effect plays an important role in nonlinear elastic waves in continuum.

In what follows, the system of basic equations and the mathematical formulation based on the derivative expansion method are given in §2. Section 3 is devoted to the derivation of the nonlinear Schrödinger equation which describes the self-modulation of the torsional mode. The known property of this equation reveals that, for the lowest dispersive branch, the modulational instability occurs for almost all wavenumber \( k \) but 2.799 < \( k \) < 3.032. In §4, a set of dynamical equations for the complex wave amplitudes of the torsional and the longitudinal modes, satisfying the second-harmonic resonance condition, will be obtained.

**§2. Formulation of the Problem**

The fundamental equations in elasticity generally consist of the conservation laws of mass, momentum, and energy together with the constitutive equation prescribing the stress-strain relation. Assuming that the wave motion takes place adiabatically, however, we need not the details of the energy equation. Thus the complete system consists of the conservation laws for mass and momentum supplemented by the constitutive equation.

The use of Lagrangian description has the advantage that the acceleration vector and the boundary condition are expressed in the linear form without any approximations. Letting \( \mathbf{u} = [u_r, u_\theta, u_z]^* \) be the displacement vector in the cylindrical system \((r, \theta, z)\), the equation of motion for a homogeneous isotropic elastic rod can be written as\cite{10,12,13}

\[
\frac{\partial^2 u_r}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r S_{rr} \right) + \frac{\partial^2 s_{rz}}{\partial z^2} - \frac{S_{\theta\theta}}{r},
\]

\[
\frac{\partial^2 u_\theta}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r S_{r\theta} \right) + \frac{\partial^2 s_{z\theta}}{\partial z^2} + \frac{S_{\theta\theta}}{r},
\]

\[
\frac{\partial^2 u_z}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left( r S_{z\theta} \right) + \frac{\partial^2 s_{zz}}{\partial z^2},
\]

where \( S_{ij} \) denotes the Lagrangian stress tensor and the \( \theta \)-dependence has been dropped, since we are concerned with the axisymmetric wave motion. It is reasonable approximation in the nonlinear problems to employ the constitutive equation, which relates the Kirchhoff’s stress tensor \( T_{ij} \) to the Lagrangian strain tensor \( E_{ij} \), in the form

\[
T_{ij} = \frac{2\sigma}{1-2\nu} I_i \delta_{ij} + 2E_{ij} + \left\{ \frac{l_1^2}{\mu} \left( \frac{2m-n}{\mu} \right) I_j \right\} \delta_{ij} + \left( \frac{2m-n}{\mu} \right) I_j E_{ij} + \frac{n}{\mu} E_{ik} E_{kj},
\]

where \( \sigma \) and \( \mu \) are, respectively, the Poisson’s ratio and one of the Lamé constants \((\lambda, \mu)\) and \( \delta_{ij} \) the Kronecker’s delta. We denote by \( l_1 \) and \( l_2 \), respectively, the first and the second invariants of the strain tensor \( E_{ij} \). The coefficients \( l, m, \) and \( n \) are the third-order elastic constants called Murnaghan constants\cite{10,14} but \( l \) disappears throughout the following analysis. In addition to this material nonlinearity, above system contains different kind of nonlinearity which comes from the nonlinear terms in \( S_{ij} \) and \( E_{ij} \) with respect to \( u \) (see Appendix A).

In the system of basic equations, all the quantities have been normalized with respect to the characteristic length \( r_0 \) (radius of the rod) and the characteristic speed \( (\mu/\rho_0)^{1/2} \) (the speed of shear wave), \( \rho_0 \) being the undisturbed density, and the suffix \( i, j, \) or \( k \) stands for either \( r, \theta, \) or \( z \). The relevant boundary conditions to the free surface are given by

\[
T_{rr} = T_{r\theta} = T_{rz} = 0 \quad \text{at} \quad r = 1.
\]

The derivative expansion method based on the multiple scales\cite{15,16} is now employed to investigate the nonlinear modulation. Following the usual procedure of the derivative expansion method, we expand \( u \) and the differential
operators into the asymptotic series:
\[
\mathbf{u} = \sum_{n=1}^{\infty} \varepsilon^n \mathbf{u}^{(n)}(r; z_0, z_1, \ldots, t_0, t_1, \ldots),
\]
where the small parameter \(\varepsilon\) measures the weakness of the nonlinearity and, at the same time, the narrowness of the sideband width of the carrier wavenumber centered around a definite wavenumber. In accordance with the expansion (4), the other field quantities can also be expanded in powers of \(\varepsilon\) and the superscript indicates its order. The multiple scales \(z_n = \varepsilon^n z\) and \(t_n = \varepsilon^n t\) \((n = 0, 1, 2, \ldots)\) are introduced to specify the slow variations of the amplitude compared with the phase of the carrier waves.

Substituting the expansions (4) and (5) into the system of basic equations (1a-c)-(3), (A·1) and (A·2) and collecting terms of the same powers in \(\varepsilon\), we obtain the successive series of the equations in the vector form:
\[
\begin{align*}
\frac{\partial^2 \mathbf{u}^{(n)}}{\partial t^2} &= \frac{2(1-\sigma)}{1-2\sigma} \text{grad div} \mathbf{u}^{(n)} - \text{rot rot} \mathbf{u}^{(n)} \\
&\quad + \psi^{(n)}(u_1^{(1)}, \ldots, u_{n-1}^{(1)}), \\
T_r^{(n)} &= T_{r \theta}^{(n)} = T_{r z}^{(n)} = 0 \quad \text{at} \quad r = 1,
\end{align*}
\]
in which the vector \(\psi^{(n)} = [\psi_r^{(n)}, \psi_{\theta}^{(n)}, \psi_z^{(n)}]\) stands for the inhomogeneous terms which are known from the preceding steps and \(\psi^{(1)} = O\).

Let us first consider the first order problem with respect to \(\varepsilon\), which is equivalent to the linear problem. Equation (6) with \(n = 1\) permits two types of propagation modes in the axisymmetric framework. One of them is the longitudinal mode which has no circumferential motion \((u_\theta^{(1)} = 0\)). The other two components of the displacement are given by \(u_r^{(1)} = U_r(r; \omega, k)e^{i\psi}\) and \(u_z^{(1)} = iU_z(r; \omega, k)e^{i\psi}\) with
\[
\mathbf{W}(\omega, k) = \begin{bmatrix}
-2\sigma(1 - 2\sigma)J_0(\omega) & -2\sigma (J_0(\omega) - J_1(\omega)) \\
-2\sigma (J_0(\omega) - J_1(\omega)) & -2k\sigma J_1(\omega)
\end{bmatrix},
\]
so that the nontrivial condition for \(R\) requires \(D_z(\omega, k)\equiv|\mathbf{W}(\omega, k)|=0\) which is the linear dispersion relation for the longitudinal mode. In the same way, \(D_i(\omega, k)\equiv|\mathbf{J}_2(\beta)|=0\) can be derived as the linear dispersion relation for the torsional mode.

The dispersion curves corresponding to each mode are plotted in Fig. 1, for instance, in the
case of Cu($\sigma=0.33$). The pairs of dots combined by the straight lines represent examples of the two modes engaged in the second-harmonic resonance. It is found that the resonance ($\omega_b=2\omega_a, k_b=2k_a$) can be classified into three types, i.e., $D_1(\omega_a, k_a)=D_1(\omega_b, k_b)=0$, $D_1(\omega_a, k_b)=D_1(\omega_b, k_a)=0$, and $D_1(\omega_a, k_a)=D_1(\omega_b, k_b)=0$. Among them only the first type will be dealt with in §4, since we are mainly concerned with the torsional mode as a fundamental mode.

§3. Self-Modulation of the Torsional Mode
(Nonresonant Case)

Considering the self-modulation of quasi-monochromatic wavetrains of the dispersive torsional mode, we choose the following plane wave solution as the starting solution:

$$
\begin{align*}
\psi_0^{(1)} &= A(z_1, z_2, \ldots, t_1, t_2, \ldots)J_1(\beta r) e^{i\phi} + \text{c.c.}, \\
\psi_r^{(1)} &= 0.
\end{align*}
$$

where $A$ is the complex amplitude, $(\omega, k)$ satisfies $D_1(\omega, k)=0$, and c.c. denotes the complex conjugate to the preceding term.

Substituting the solution (12) into the system of the basic equations and rewriting them in the vector form (6), it follows that

$$
\psi^{(2)} = \begin{bmatrix} F_t^{(2)}(r) \\ 0 \\ iF_z^{(2)}(r) \end{bmatrix} e^{i\phi} + \begin{bmatrix} 0 \\ 2i\omega \left( \frac{\partial A}{\partial t_1} + V_{gt} \frac{\partial A}{\partial z_1} \right) J_1(\beta r) e^{i\phi} + \text{c.c.} + \begin{bmatrix} 0 \\ F_\psi^{(2)}(r) \end{bmatrix} |A|^2,
\end{align*}
$$

in which $V_{gt}(=\omega_0/dk=k/\omega)$ is the group velocity of the torsional mode. The explicit forms of the $F$'s are too complicated to be presented so as to save space.

The $\theta$-component of the eq. (6) in the second order problem is decoupled and the solution for $u_\theta^{(2)}$ can be easily obtained as

$$
\begin{align*}
u_\theta^{(2)} &= i\pi \omega \left( \frac{\partial A}{\partial t_1} + V_{gt} \frac{\partial A}{\partial z_1} \right) \left[ J_1(\beta r) \int_0^r rJ_1(\beta r) N_1(\beta r) dr - N_1(\beta r) \int_0^r rJ_1^2(\beta r) dr \right] + \text{c.c.}
\end{align*}
$$

Using the boundary condition $T_{z_2}^{(2)}(r)(\partial \psi^{(2)}/\partial r)=0$ at $r=1$ and the dispersion relation $J_2(\beta)=0$, we obtain

$$
\frac{\partial A}{\partial t_1} + V_{gt} \frac{\partial A}{\partial z_1} = 0,
$$

so that $u_\theta^{(2)}=0$. This result implies that, in the lowest order of approximation, the complex amplitude $A$ is kept constant along the characteristic moving with the group velocity $V_{gt}$.

Next, in order to obtain the components $u_t^{(2)}$ and $u_z^{(2)}$, we must solve the inhomogeneous equations with the harmonic part ($\propto A^2 e^{i\phi}$) and the steady part ($\propto |A|^2$), which are interpreted as the forcing terms of $r$ and $z$-directions generated by the fundamental torsional mode due to the nonlinearity. When it is assumed that

$$
\begin{align*}
u_t^{(2)} &= G_t^{(2)}(r) A^2 e^{i\phi} + \text{c.c.} + G_\psi^{(2)}(r) |A|^2, \\
u_z^{(2)} &= iG_z^{(2)}(r) A^2 e^{i\phi} + \text{c.c.,}
\end{align*}
$$

the steady part $G_t^{(2)}$ may be directly integrated to give

$$
G_\psi^{(2)} - \frac{1-2\sigma}{2(1-\sigma)} \left[ \frac{1}{r} \int_0^r \left[ \int_0^r F_\psi^{(2)}(r) dr \right] dr \right] = 0.
$$

For the harmonic part, the substitution of (16a, b) into eq. (6) with inhomogeneous terms (13) leads to

$$
\begin{align*}
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + 4\omega^2 \right) A &= -\frac{1-2\sigma}{2(1-\sigma)} \left[ \frac{1}{r} \frac{d}{dr} (rF_t^{(2)}(r)) - 2kF_z^{(2)}(r) \right], \\
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + 4\beta^2 - \frac{1}{r^2} \right) \Omega_\theta &= -2kF_t^{(2)}(r) + \frac{d}{dr} F_z^{(2)}(r).
\end{align*}
$$
with
\[ A = \frac{1}{r} \frac{d}{dr} (rG_r^{(2)}) - 2kG_z^{(2)}, \tag{20a} \]
\[ \Omega_\theta = 2kG_r^{(2)} - \frac{d}{dr} G_z^{(2)}. \tag{20b} \]

These equations can be immediately integrated by the method of variation of constants and then their particular solutions are given as
\[ \Delta = \frac{1-2\sigma}{2(1-\sigma)} \frac{\pi}{2} \begin{cases} J_0(2\sigma r) \int_{0}^{r} [2\sigma F_r^{(2)}(r)J_1(2\sigma r) - 2kF_z^{(2)}(r)J_0(2\sigma r)] \, dr \\ - N_0(2\sigma r) \int_{0}^{r} [2\sigma F_z^{(2)}(r)J_1(2\sigma r) - 2kF_r^{(2)}(r)J_0(2\sigma r)] \, dr \end{cases}, \tag{21a} \]
\[ \Omega_\theta = \frac{\pi}{2} \begin{cases} J_1(2\beta r) \int_{0}^{r} [2\beta F_r^{(2)}(r)N_1(2\beta r) + 2\beta F_z^{(2)}(r)N_0(2\beta r)] \, dr \\ - N_1(2\beta r) \int_{0}^{r} [2\beta F_z^{(2)}(r)J_1(2\beta r) + 2\beta F_r^{(2)}(r)J_0(2\beta r)] \, dr \end{cases}. \tag{21b} \]

From the relations (20a, b), $G_r^{(2)}$ and $G_z^{(2)}$ can be obtained as follows:
\[ G_r^{(2)} = \left( \frac{d}{dr} G_z^{(2)} + \Omega_\theta \right) \sqrt{2k}, \tag{22a} \]
\[ G_z^{(2)} = 2k \left( I_0(2kr) \int_{0}^{r} [r\Delta K_0(2kr) - r\Omega_\theta K_1(2kr)] \, dr \\ - K_0(2kr) \int_{0}^{r} [r\Delta I_0(2kr) + r\Omega_\theta I_1(2kr)] \, dr \right). \tag{22b} \]

Since the solutions just obtained are the particular solutions, the general solutions should be completed by adding the homogeneous ones $a_0^{(2)}r$, $U_r(r; 2\omega, 2k)$, and $U_z(r; 2\omega, 2k)$ to $G_r^{(2)}$, $G_z^{(2)}$, and $G_z^{(2)}$, respectively.

In order that the second order problem should be completed, we have only to determine the arbitrary constants $a_0^{(2)}$ and $R^{(2)}$ in the homogeneous solutions. The inhomogeneous parts $T_r^{(2)}$ and $iT_z^{(2)}$ in $T_r^{(2)}$ and $T_z^{(2)}$ take the form: $[\tilde{T}_r^{(2)}, \tilde{T}_z^{(2)}] = T^{(2)} A^2 e^{2i\psi} + c.c. + \tilde{T}_0^{(2)} A^2$, with $\tilde{T}^{(2)} = [T_r^{(2)}, T_z^{(2)}]$ and $\tilde{T}_0^{(2)} = [\tilde{T}_r^{(2)}, 0]$. Thus the boundary condition yields
\[ a_0^{(2)} = \left( \sigma - \frac{1}{2} \right) \left( \frac{\pi}{2} T_r^{(2)}(1) - \int_{0}^{1} F_\phi^{(2)}(r) \, dr - 2G_\phi^{(2)}(1) \right), \tag{23} \]
\[ W(2\omega, 2k)R^{(2)} + T^{(2)} = 0, \tag{24a} \]
with
\[ T^{(2)} = \tilde{T}^{(2)}(1) + \left[ \frac{2\sigma}{1-2\sigma} A + \frac{d}{dr} G_r^{(2)} \right] \bigg|_{r=1}^{r=0} \left( \frac{2kG_r^{(2)} + \frac{d}{dr} G_z^{(2)} \bigg|_{r=1}^{r=0}}{1} \right), \tag{24b} \]
from which one gets $R^{(2)} = -W^{-1}(2\omega, 2k)T^{(2)}$ unless the matrix $W(2\omega, 2k)$ is singular. The matrix $W(2\omega, 2k)$ may become singular if the fundamental torsional mode drives the longitudinal mode with $(2\omega, 2k)$ to cause the second-harmonic resonance between them. However, at present, we exclude this resonant case, leaving it to the next section. Thus we can complete the second order solution as $u_\theta^{(2)} = 0$ and
\[ \begin{bmatrix} u_r^{(2)} \\ u_z^{(2)} \end{bmatrix} = \begin{bmatrix} U_r(r; 2\omega, 2k) + G_r^{(2)}(r) \\ i[U_z(r; 2\omega, 2k) + G_z^{(2)}(r)] \end{bmatrix} A^2 e^{2i\psi} + c.c. + \begin{bmatrix} a_0^{(2)} + G_\phi^{(2)}(r) \\ 0 \end{bmatrix} |A|^2. \tag{25} \]
We now proceed to the third order problem, where only the \( \theta \)-component of eq. (6) is concerned. On the substitution of solutions (12) and (25) into the system of basic equations, it follows that

\[
v_\theta^{(3)} = \left[ 2i\omega \left( \frac{\partial A}{\partial t_2} + V_{\theta t_2} \frac{\partial A}{\partial z_2} \right) + \frac{\beta^2}{\omega^2} \frac{\partial^2 A}{\partial z_1^2} \right] J_1(\beta r) + F_\theta^{(3)}(r)A^2A \right) e^{i\varphi} \text{ terms in } (A^3 e^{3i\varphi}) + \text{ c.c.}
\]  

(26)

where eq. (15) has been used.

After the similar procedure to that for the second order problem, the solution \( u_\theta^{(3)} = G_\theta^{(3)}(r)e^{i\varphi} + \text{ c.c.} \) can be found as

\[
G_\theta^{(3)} = \left[ 2i\omega \left( \frac{\partial A}{\partial t_2} + V_{\theta t_2} \frac{\partial A}{\partial z_2} \right) + \frac{\beta^2}{\omega^2} \frac{\partial^2 A}{\partial z_1^2} \right] \frac{\pi}{2} \int_0^r J_1(\beta r) r F_\theta^{(3)}(r)N_1(\beta r) dr - N_1(\beta r) \int_0^r r J_1^2(\beta r) dr \\
+ |A|^2 A^2 \left[ \frac{\pi}{2} J_1(\beta r) \int_0^r r F_\theta^{(3)}(r)N_1(\beta r) dr - N_1(\beta r) \int_0^r r F_\theta^{(3)}(r)J_1(\beta r) dr \right]
\]  

(27)

Using the boundary condition \( T_\theta^{(3)} = 0 \) at \( r=1 \) and transfroming the independent variables into

\[
\xi = z_1 - V_{\theta t_1} = e^{-1}(z_2 - V_{\theta t_2}), \quad \tau = t_2,
\]  

(28)

we obtain the nonlinear Schrödinger equation:

\[
\frac{\partial A}{\partial \tau} + P \frac{\partial^2 A}{\partial \xi^2} = Q|A|^2A,
\]

(29)

where the real coefficients \( P \) and \( Q \) are defined by

\[
P = \frac{1}{2} \frac{d^2 \omega}{dk^2} = \frac{\beta^2}{2\omega^3} > 0,
\]

(30)

\[
Q = -[\omega J_1^2(\beta)]^{-1} \int_0^1 r F_\theta^{(3)}(r)J_1(\beta r) dr + \left[ \frac{\pi}{2} \omega \beta N_2(\beta) \right]^{-1} \times \left[ \frac{d}{dr} \left[ (U_\omega(r; 2\omega, 2k) + G_\omega^{(2)}(r) + G_\omega^{(2)}(r)) / J_1(\beta r) \right] - \frac{n k}{2\mu J_1(\beta)} \times 2k(U_\omega(r; 2\omega, 2k) + G_\omega^{(2)}(r)) + \frac{d}{dr} \left( U_\omega(r; 2\omega, 2k) + G_\omega^{(2)}(r) + \frac{k^2}{4} J_1^2(\beta r) \right) \right]_{r=1}
\]

(31)

As is known from the existing work,\(^2,3,16-18\) eq. (29) has the steady-state solutions which generally represent wavetrains expressible in terms of the Jacobian elliptic functions. They include a bright and a dark envelope soliton, a phase jump, and a plane wave with constant amplitude as special cases.

But even more, the plane wave solution is known to be modulationally unstable, if \( PQ < 0 \). Figure 2 shows the calculated result of the product \( PQ \) versus wavenumber \( k \) for the lowest dispersive branch (\( \beta = \beta_1 \)) in the case of Cu (\( \mu = 4.77 \times 10^3 \) kgw/mm\(^2\); \( m = -6.2 \times 10^4 \) kgw/mm\(^2\), \( n = -15.9 \times 10^4 \) kgw/mm\(^2\)).\(^10\) It is therefore concluded that, for this branch, the torsional wavetrains of constant amplitude are modulationally unstable for almost all \( k \) but 2.799 < \( k < 3.032 \). Moreover we carried out the calculation of \( PQ \) vs \( k \) for \( \beta = \beta_2 \) and \( \beta = \beta_3 \) (see Fig. 1) and had the same results in substance as \( \beta = \beta_1 \), i.e., \( PQ < 0 \) for smaller values of \( k \) than the smallest critical wavenumber, \( PQ \rightarrow \pm \infty \) on both sides of the critical ones.
and \( PQ \to -\infty \) as \( k \to \infty \).

**§4. Resonant Case (the Second-Harmonic Resonance with Longitudinal Mode)**

We shall now investigate the resonant case noted in the preceding section. In this case, the longitudinal mode with \((2\omega, 2k)\) can propagate simultaneously with the torsional mode with \((\omega, k)\). Therefore, in order to render the solution uniformly valid, it is necessary to include the longitudinal mode of \((2\omega, 2k)\) in the first order solution in addition to the torsional mode of \((\omega, k)\).

Thus the starting solution we use here is

\[
\mathbf{v}^{(2)} = \begin{bmatrix}
\{ i \left[ 4\omega U_r \frac{\partial B}{\partial t_1} + \left( 2k U_r + \frac{d}{dr} U_z \right) \frac{\partial B}{\partial z_1} \right] + F_r(r) A^2 \} e^{2i\phi} \\
\{ 2i\omega \left( \frac{\partial A}{\partial t_1} + V_{g1} \frac{\partial A}{\partial z_1} \right) J_1(\beta r) + F^{(2)}(r) \bar{A} \} e^{i\phi} \\
\{ -4\omega U_z \frac{\partial B}{\partial t_1} + \left[ \frac{2\sigma - 1}{1 - 2\sigma} \frac{1}{r} \frac{d}{dr} (rU_r) - \frac{2(1 - \sigma)}{1 - 2\sigma} \cdot 2k U_z \right] \frac{\partial B}{\partial z_1} + iF^{(2)}(r) A^2 \} e^{2i\phi} \}
\end{bmatrix} + \text{terms in } (e^{4i\phi}, e^{3i\phi}, \text{and constant in } \psi) + \text{c.c.,}
\]

where the shortened notations, \( U_r \) and \( U_z \), are used in place of \( U_r(r; 2\omega, 2k) \) and \( U_z(r; 2\omega, 2k) \), respectively.

Making use of the similar procedure to that in the preceding section, we can get the solution for \( \mathbf{v}^{(2)} \) after tedious calculations. The boundary conditions for \( T_r^{(2)} \) and \( T_z^{(2)} \) may be arranged into the form

\[
\mathbf{W}(2\omega, 2k) R^{(2)} + i \frac{\partial B}{\partial t_1} \frac{\partial \mathbf{W}(2\omega, 2k)}{\partial (2\omega)} R^{(1)} = i \frac{\partial B}{\partial z_1} \frac{\partial \mathbf{W}(2\omega, 2k)}{\partial (2k)} R^{(1)} - i \mathbf{W}(2\omega, 2k) S + T^{(2)} A^2 = 0,
\]

where the vector \( T^{(2)} \) denotes, as before, the coefficients of \( A^2 e^{2i\phi} \) in the stress components \( T_r^{(2)} \) and \( T_z^{(2)} \) and the explicit form of the vector \( S \) is found in Appendix B. Since the matrix \( \mathbf{W}(2\omega, 2k) \) is now singular as remarked, we must require the compatibility condition, which is obtained by multiplying eq. (34) by such left eigenvector \( L^{(1)} \) as \( L^{(1)} \mathbf{W}(2\omega, 2k) = 0 \), form the left.

From the boundary condition for \( T_r^{(2)} \) and the compatibility condition, we have, respectively,

\[
\frac{\partial A}{\partial t_1} + V_{g1} \frac{\partial A}{\partial z_1} = i\gamma_1 \bar{A} B,
\]

\[
\frac{\partial B}{\partial t_1} + V_{g1} \frac{\partial B}{\partial z_1} = i\gamma_1 A^2,
\]

where \( V_{g1} \) is the group velocity of the longitudinal mode defined by

\[
V_{g1} \equiv \frac{dk}{d\omega} = -L^{(1)} \frac{\partial \mathbf{W}(2\omega, 2k)}{\partial (2\omega)} R^{(1)} \left[ L^{(1)} \frac{\partial \mathbf{W}(2\omega, 2k)}{\partial (2k)} R^{(1)} \right],
\]

and the real coefficients \( \gamma_1 \) and \( \gamma_1 \) are expressed as

\[
\gamma_1 = -[\omega J_1^2(\beta)]^{-1} \int_0^1 r F_r^{(2)}(r) J_1(\beta r) dr + \left[ \frac{\pi}{4} \omega \beta N_2(\beta) \right]^{-1} \left\{ \frac{1}{2} \frac{d}{dr} [U_z(r; 2\omega, 2k) J_1(\beta r)] \\ - \frac{nk}{4\mu J_1(\beta)} \left[ 2k U_x(r; 2\omega, 2k) + \frac{d}{dr} U_x(r; 2\omega, 2k) \right] \right\}_{r=1},
\]

(37a)
\[ \gamma_1 = L^{(1)} \cdot T^{(2)} \left[ L^{(1)} \frac{\partial W(2\omega, 2k)}{\partial (2\omega)} \cdot R^{(1)} \right]. \]

This is the set of dynamical equations which governs the slow evolution of the coupled two modes in resonance.

To the dynamical equations \((35a, b)\), the steady-state solutions have been obtained,\(^7\) which can be also, in general, expressed in terms of the Jacobian elliptic functions. In a particular case, the amplitude \(|A|\) represents a bright envelope soliton while \(|B|\) represents a dark envelope soliton.

By means of the numerical calculation, it is evaluated, for the critical wavenumber \(k = k_1 = 2.166\) (see Fig. 1), that \(V_{g1} = 0.3886\), \(V_{g1} = 0.6697\), and \(\gamma_1 = 2.705\), \(\gamma_1 = 1.324\).

Reverting to the original variables \((z, t)\) in eqs. \((35a, b)\), the magnitude of the nonlinear terms is found to be of the order of \(\varepsilon\) compared with the linear terms. Therefore the nonlinearity comes to prevent the independent wave propagation in the long spatial and the slow temporal scales, i.e., \(z_1\) and \(t_1\). It may be directly verified, for the special cases for which \(|A| < |B|\) or vice versa, that the smaller amplitude begins to grow monotonously owing to the energy supply from the other.

**§5. Summary**

It has been shown that the derivative expansion method, one of the typical singular perturbation method, is available for the weakly nonlinear dispersive waves in elasticity, as well as in other dispersive systems. The analysis of the nonlinear modulation of the torsional mode in an elastic rod was divided into two parts, i.e., nonresonant and resonant cases; for the former the governing equations can be reduced to the nonlinear Schrödinger equation and for the latter to a set of dynamical equations. The second-harmonic resonance discussed in the present paper differs from the others\(^6,7\) in the point that the different modes, torsional and longitudinal, are concerned in the present problem. Thus one mode may excite the other mode sharing the wave energy. Another resonant case, the third-harmonic resonance, can also take place provided that the dispersion relation \(D_3(3\omega, 3k) = 0\) does hold. In such cases, the third order problem in §3 should be modified. Nayfeh\(^20\) studied this type of resonance for capillary-gravity waves on deep water.

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**Appendix A**

The relation between the Lagrangian stress tensor \(S_{ij}(\neq S_{ji})\) and the Kirchhoff’s stress tensor \(T_{ij}(= T_{ji})\):

\[
\begin{align*}
S_{rr} & \quad S_{r\theta} & \quad S_{rz} \\
S_{\theta r} & \quad S_{\theta \theta} & \quad S_{\theta z} \\
S_{z r} & \quad S_{z \theta} & \quad S_{zz}
\end{align*}
\begin{bmatrix}
1 + \frac{\partial u_z}{\partial r} & -\frac{u_\theta}{r} & \frac{\partial u_z}{\partial z} \\
\frac{\partial u_\theta}{\partial r} & 1 + \frac{u_z}{r} & \frac{\partial u_\theta}{\partial z} \\
\frac{\partial u_z}{\partial r} & 0 & 1 + \frac{\partial u_z}{\partial z}
\end{bmatrix}
\begin{bmatrix}
T_{rr} \\
T_{\theta r} \\
T_{z r}
\end{bmatrix}
\begin{bmatrix}
T_{r\theta} \\
T_{\theta \theta} \\
T_{z \theta}
\end{bmatrix}
\begin{bmatrix}
T_{rz} \\
T_{r z} \\
T_{z z}
\end{bmatrix}
\tag{A.1}
\end{align*}
\]

Some writers call \(S_{ij}\) and \(T_{ij}\), respectively, the first and the second Piola-Kirchhoff’s stress tensors.\(^{10}\)

The Lagrangian strain tensor \(E_{ij}\):

\[
2E_{rr} = 2\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2,
\]

\[
2E_{r\theta} = 2\frac{u_\theta}{r} + \left(\frac{u_\theta}{r}\right)^2 + \left(\frac{u_z}{r}\right)^2,
\]

\[
2E_{z r} = 2\frac{u_z}{r} + \left(\frac{u_z}{r}\right)^2 + \left(\frac{u_\theta}{r}\right)^2.
\]
\[2E_{zz} = 2 \frac{\partial u_z}{\partial z} + \left( \frac{\partial u_y}{\partial z} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \]

\[2E_{\theta \theta} = r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) - \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial r} \left( \frac{u_r}{r} \right), \]

\[2E_{\theta z} = \frac{\partial u_\theta}{\partial z} - \frac{\partial u_z}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{\partial u_\theta}{\partial z} \frac{u_r}{r}, \]

\[2E_{z r} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} + \frac{\partial u_\theta}{\partial r} \frac{u_r}{r} + \frac{\partial u_\theta}{\partial z} \frac{u_r}{r} + \frac{\partial u_r}{\partial z} \frac{u_\theta}{r} + \frac{\partial u_z}{\partial z} \frac{u_\theta}{r}. \]

**Appendix B**

The explicit representation for the vector \( S \):

\[
S = \begin{pmatrix} a^{(1)} \frac{\partial x}{\partial \omega} \frac{\partial B}{\partial t_1} - \frac{\partial x}{\partial k} \frac{\partial B}{\partial z_1} H_0(2\lambda) \\ b^{(1)} \frac{\partial \beta}{\partial \omega} \frac{\partial B}{\partial t_1} - \frac{\partial \beta}{\partial k} \frac{\partial B}{\partial z_1} H_1(2\lambda) \end{pmatrix},
\]

where

\[
H_\mu(v) = \frac{J_\mu(v)}{J_\mu^2(v)} + \int_0^1 \frac{1}{vr} \left[ 1 - \frac{J_{\mu-1}(vr)J_{\mu+1}(vr)}{J_\mu^2(vr)} \right] dr.
\]

**References**