Shock Wave Propagation in a Viscoelastic Rod

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Summary
The propagation of torsional and longitudinal shock waves in a thin circular viscoelastic rod is investigated theoretically. Assuming a "nearly elastic" rod, new approximate governing equations are derived for the respective mode. Two typical relaxation functions are compared. One is the exponential function type well known as Maxwell-Voigt model. The other is the power function type newly proposed here as a model having a continuous relaxation spectrum. Based on the approximate equations, the steady shock conditions and profiles are shown for both types. Also the unsteady unidirectional wave propagation is briefly discussed.

Introduction
Recently nonlinear viscoelastic wave propagation has attracted much attention with the diverse applications of polymers [1]. The propagation of shock waves is one of the main features in contrast to the linear wave propagation. It results from a balance between the nonlinearity and the dissipation. The viscoelastic materials are not only dissipative but dispersive. In particular, if we consider a body with a boundary, there appears another dispersion which is attributed to the geometrical configuration of the body and is distinguished from the material dispersion. Most previous works are concerned with a wave propagation in an infinite body without geometrical dispersion and little has been made for a body with a boundary. One of the main purposes of this paper is to consider, as a fundamental example, the propagation of shock waves in a thin circular viscoelastic rod. The other is concerned with the relaxation character of viscoelastic materials. The viscoelastic behavior is often modeled by a combination of several springs and dashpots [2]. Such models have succeeded in explaining intuitively the inner relaxation mechanism. But they are inadequate in describing the actual viscoelastic behavior[3].
It is usually characterized by infinitely many relaxation times, whereas the spring-dashpot model has only finite ones. In other words, the continuous relaxation spectrum is required in place of the discrete one characterizing the spring-dashpot model. Here we propose, as a plausible model, a new relaxation function of power function type, whose relaxation spectrum has a continuous distribution approximated by a power law.

In what follows, we consider the torsional and longitudinal shock waves, respectively. We remark that since the torsional waves concerned here are the lowest mode, they are geometrically non-dispersive in spite of the presence of the boundary. For comparison, Maxwell-Voigt model is considered in addition to this power function type. The shock conditions and the explicit shock profiles are obtained for both types. Finally for the unsteady unidirectional wave propagation, the reduction of the approximate equations is briefly discussed.

**Approximate Governing Equations**

The approximate governing equations for the torsional and the longitudinal waves are presented. Their derivation should be referred to [4,5], but is briefly outlined here. Based on the three dimensional nonlinear theory of viscoelasticity in Lagrangian formulation, the boundary value problem is posed at a free lateral surface of an infinite rod. With the thermal effect ignored, a "nearly elastic" rod is assumed in which the dependence of stress on a history of strain is weak. In the constitutive equations, the elastic response is taken up to the third order in strain, while the viscoelastic one is taken in a form of the linear hereditary integral. In derivation, we make use of three small parameters $\varepsilon$, $\delta$, and $\gamma$, $(0<\varepsilon, \delta, \gamma<<1)$ designating, respectively, the thinness of the rod, the magnitude of displacement, and the weakness of viscoelasticity, a lengthscale being normalized by a wavelength. By the assumption of thin rod, the displacement is sought in a power series of the radial coordinate. The approximate governing equations are derived so that the boundary conditions may be fulfilled.

For the torsional waves, the finite torsional deformation couples with the induced longitudinal one. Taking account of both geometrical and material nonlinearity, we have the coupled equations:
\[ \phi_{tt} - G\phi_{zz} - \gamma \int_{-\infty}^{t} K_T(t-t') \phi_{ztz} dt' = -2\varepsilon^2 [ (\psi \phi_z)_t - G(\psi \phi_z) ]_z \]
\[ + \varepsilon^2 (a \omega_z + b_1 \phi_t^2 + b_2 \phi_z^2 \phi_z)_z, \] (1)

and \[ w_{tt} - Ew_{zz} = (c_1 \phi_t^2 + c_2 \phi_z^2)_z, \] (2)

with \[ \psi = -\sigma w_z + d_1 \phi_t^2 + d_2 \phi_z^2, \] (3)

where \( \phi \) and \( w \) represent the finite angle of torsion and the longitudinal displacement, respectively, and the suffixes \( t \) and \( z \) imply partial differentiation with respect to \( t \) and \( z \); \( t \) being time and \( z \) the axial coordinate. Here \( G, E, \) and \( \sigma \) denote the modulus of rigidity, Young's modulus, and Poisson's ratio, respectively, and \( K_T \) denotes the shear stress relaxation function. The constants \( a, b_i, c_i, \) and \( d_i (i=1,2) \) are determined by \( G, E, \) and the higher order (equilibrium) elastic constants.

For the longitudinal waves, on the other hand, account is taken of not only the finite deformation but also the lateral contraction of the rod. Then we have

\[ w_{tt} - Ew_{zz} - (\varepsilon \sigma)^2 w_{ttzz}/2 + \gamma \int_{-\infty}^{t} K_L(t-t') w_{ztz} dt' = \delta(\omega_z)_z^2, \] (4)

where \( K_L \) denotes the tensile stress relaxation function and \( \delta \) is a constant given by the elastic constants.

Relaxation Functions and Linear Dispersion Relations

The viscoelastic behavior of rod is specified by the relaxation functions \( K_T \) and \( K_L \). For the spring-dashpot model, they can be expressed by a sum of several exponential functions including the delta or step function as a limiting case:

\[ K_{T,L}(t) = \sum_{i} K_{T,L}^{(i)} \exp(-t/T_i), \] (5)

where \( T_i \)'s imply the relaxation times and \( K_{T,L}^{(i)} \)'s are constants.

But in order to describe actual viscoelastic materials that have infinitely many relaxation times, the relation (5) should be extended as

\[ K_{T,L}(t) = \int_{-\infty}^{\infty} H_{T,L}(T') \exp(-t/T') d(lnT'), \] (6)

where \( H_{T,L}(T') \) are relaxation spectra. Here we consider two simple but typical types for \( K_T \) and \( K_L \). One is the exponential function type (Type I) and the other the power function type (Type II).

Type I: \[ K_{T,L}(t) = \exp(-\kappa t), \quad (\kappa > 0), \] (7)
Type II: \( K_{T,L}(t) = t^{-\nu}, \) (0<\(\nu<1\)), \( \) (8)

where Type I has a single relaxation time \( \kappa^{-1} \) in (5) and Type II is a model in which \( H_{T,L}(T') = T'^{-\nu}/\Gamma(\nu) \), \( \Gamma(\nu) \) being the gamma function.

For both types, we note the linear dispersion relations. For the torsional waves, the phase velocity \( c \) is given by \( c^2 = G + \gamma/(1 + i\kappa/\omega) \) for Type I and \( c^2 = G + \gamma \Gamma(1 - \nu)(-i\omega)^\nu \) for Type II, \( \omega \) being a frequency. For Type I, we remark two characteristic sound speeds, the instantaneous sound speed \( c_i = (G + \gamma)^{1/2} \) as \( \omega \to \infty \) and the equilibrium one \( c_e = G^{1/2} \) as \( \omega \to 0 \). But for Type II, \( c_i \) becomes infinite because of the infinite stiffness at \( t=0 \), though \( c_e = G^{1/2} \). For the longitudinal waves, \( c \) is given by \( c^2 = \epsilon - (\epsilon \sigma)^2/2 + \gamma/(1 + i\kappa/\omega) \) for Type I and \( c^2 = \epsilon - (\epsilon \sigma)^2/2 + \gamma \Gamma(1 - \nu)(-i\omega)^\nu \) for Type II. The geometrical dispersion becomes important as \( \omega \to \infty \).

**Steady Shock Waves**

We investigate the steady shock wave propagation into the unstrained state far ahead \( (z \to \infty) \) with an equilibrium strain state far behind \( (z \to -\infty) \). Putting \( \eta = t - z/\lambda, \lambda > 0 \) being a shock velocity and retaining only the lowest order terms in \( \epsilon^2, \delta, \) and \( \gamma \), eqs. (1)-(3) and eq. (4) are reduced, respectively, to

\[
U \Phi - \text{sgn}\alpha \Phi^2 = \int_{-\infty}^{\eta} K_T(\eta - \eta') \phi_{\eta'}, d\eta',
\]

(9)

with \( U = (\lambda^2 - G)/\gamma \) and \( \phi = (\epsilon^2 \alpha^2/\gamma)^{1/2} d\phi/d\eta \), where \( \alpha = [a(c_1 G + c_2)/(G - E) + b_1 G + b_2]/G \), and

\[
V W - W^2 - \mu W_{\eta \eta} = \int_{-\infty}^{\eta} K_L(\eta - \eta') W_{\eta'}, d\eta',
\]

(10)

where \( V = (\lambda^2 - E)/\gamma, W = -\delta \mu/(\gamma \lambda) d\omega/d\eta, \) and \( \mu = (\epsilon \sigma)^2/(2\gamma) \). We note that for the same \( W \), the difference in sign of \( \beta \) produces physically compression or expansion.

By examining the necessary conditions for the shock waves to exist, it is found for the torsional waves that \( \alpha \) and \( U \) should be positive and that the equilibrium value at \( \eta = \infty \) is given by \( \Phi^2 = U \). For the longitudinal waves, it is found that \( V \) should be positive and that the equilibrium value is given by \( W = V \). Thus from \( U, V > 0 \), each shock velocity is always greater than the respective equilibrium sound speed \( G^{1/2} \) and \( E^{1/2} \). The velocity becomes faster as the shock strength \( \Phi_\infty \) and \( W_\infty \) increase. Although these conditions are de-
rived on assuming smooth solutions, we remark the existence of a discontinuous solution in $\phi$. For Type I and $\alpha>0$, eq.(9) allows a discontinuity with a jump $\phi_0^2=U-1$. For Type II, no discontinuity is allowed. In eq.(10), on the other hand, $\mu \eta \eta$ prevents an occurrence of a discontinuity, which should be compared with eq.(9). For Type I, eqs.(9) and (10) can be rewritten in the nonlinear differential equations, which are solved analytically and numerically. For Type II, however, they remain the nonlinear and singular integral equations, which are solved by the numerical quadrature.

**Torsional Shock Waves**

For Types I and II, the typical shock profiles are displayed in Figs.1 and 2, respectively. For Type I, there is the critical speed $(G + \gamma)^{1/5} (U=1)$ beyond which the shock profiles change remarkably. For a velocity below it $(0<U<1)$, a smooth and monotonous transition appears, while beyond it $(U>1)$, a discontinuity followed by a monotonous profile appears. The discontinuity is introduced according to the jump condition $\phi_0^2=U-1$. For Type II, the profiles are always smooth and monotonous. The step-up is exponential but there appears a very slow relaxation region in a trail. This tendency becomes prominent as $\nu$ decreases. Thus no sharp shock layer can be seen in Type II, which should be compared with the result in Type I.

**Longitudinal Shock Waves**

For Types I and II, the typical profiles are shown, respectively, in Figs.3 and 4. The geometrical dispersion gives rise to oscillatory profiles. This is the essential difference from the torsional waves. For Type I, the shock waves step up exponentially, then overshoot the equilibrium value, and approach it exponen-
tially with an oscillation. As \( \kappa^2 \mu \) decreases, the step-up becomes steep while as it increases, a strong oscillation appears. For a weak shock wave, the profiles are monotonous for \( \kappa^2 \mu \geq 4 \) and oscillatory for \( 0 < \kappa^2 \mu \leq 4 \). For Type II, there also appears a very wide oscillatory relaxation region. Except for large \( V \), the profiles do not overshoot the equilibrium value but undulate below it. The tendency with respect to \( \mu \) is similar to that in Type I. We only note that for the same \( V \), a pronounced oscillation appears as \( \nu \) decreases.

**Unsteady Unidirectional Waves**

For unsteady unidirectional waves into the unstrained state, eqs. (1)-(4) are relatively simplified. Introducing the new coordinates \( \tau = \varepsilon^2 z / G^{\frac{3}{5}} \) and \( \xi = t - z / G^{\frac{3}{5}} \), eqs. (1)-(3) are reduced to

\[
f_\xi - 3 \alpha f^2 f_\xi / 2 = \gamma / (2 \varepsilon^2 G) \int \xi_0^{\xi} K_\tau (\xi - \xi') f_{\xi \xi'} d\xi',
\]

where \( f = \phi_z \). For Type I, eq. (11) can be rewritten in the differential equation. If, in particular, the rapid relaxation is assumed, \( \kappa^{-1} - \varepsilon^2 / \gamma \ll 1 \), then "cubic Burgers' equation" is reduced with the right hand side of (11) replaced by \( \gamma f_{\xi \xi} / (2 \varepsilon^2 G \kappa) \). For Type II, eq. (11) is interpreted as "generalized cubic Burgers' equation" by introducing the fractional derivative defined by

\[
\int_{-\infty}^{\xi} \frac{1}{(\xi - \xi')^{1-\nu}} \frac{\partial f}{\partial \xi'} d\xi' \equiv \Gamma(1-\nu) \frac{\partial^\nu f}{\partial \xi^\nu}, \quad (0 < \nu < 1).
\]

The similar reduction can be made for eq. (4). For the rapid relaxation in Type I, the usual K-dV-Burgers' equation is reduced. For Type II, we have the "generalized K-dV-Burgers' equation".

**References**