Reflection and Transmission of a Shallow-Water Soliton over a Barrier

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This paper deals with a shallow-water soliton incident upon a submerged barrier. Introducing an ‘edge-layer’ near the barrier, the matched-asymptotic expansion method is applied to derive uniformly valid solutions. Assuming Boussinesq equations are valid for the shallow-water regions, two ‘reduced’ boundary conditions are derived from the matching conditions. This boundary-value problem is further reduced to an ‘initial value’ problem for K-dV equations, which is solved to investigate reflection and transmission of an incident soliton. For a moderate height of the barrier, the soliton can pass over the barrier as if it were almost transparent. As the barrier becomes high, the transmitted soliton is suppressed, while the reflected wave is enhanced and evolves into one soliton with slowly decaying ripples behind.

§1. Introduction

This paper develops an edge-layer theory for a shallow-water soliton incident upon a barrier to discuss its resulting reflection and transmission. The barrier concerned here is a submerged vertical wall of infinitely thin thickness shown in Fig. 1. In reality, this models a submerged ridge or breakwater whose horizontal spread is small compared with a typical wavelength. A vertical wall with finite horizontal thickness may be regarded as a superposition of two (upward and downward) steps. If the whole water-layer is divided into three parts, i.e., the right- and left-hand sides of the barrier and the mid part over the barrier, the well-known Lamb’s conditions\(^1\) could be applied at each step to the lowest order of approximation. Taking the limit of infinitely thin thickness, however, Lamb’s conditions predict no reflection at all for any height of the barrier so that the incident wave can pass over it as if the barrier were transparent.\(^2\)

This paradoxical result stems from the use of shallow-water approximation throughout the whole region. Actually, in the vicinity of the barrier, there appears a region in which the

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Fig. 1. Definition sketch of the problem.

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vertical acceleration of the fluid becomes significant so that the shallow-water approximation breaks down locally. This is the very region that we have called an 'edge-layer' in the case of a beach\(^5\) and a stepped bottom.\(^6\)

In the present case as well, the horizontal scale of the edge-layer is comparable with the depth-scale but it is sufficiently small compared with a characteristic wavelength in the outer shallow-water regions.

Utilizing this significant scale-difference, the matched-asymptotic expansion method is successfully applied to derive uniformly valid solutions. In the shallow-water regions extending on the outside of the edge-layer, we assume Boussinesq's theory which takes account of the effects of weak nonlinearity and dispersion.\(^5\)\(^6\) From the matching conditions, we can derive two 'reduced' boundary conditions which reflect essential effects of the presence of the barrier on Boussinesq equations. Upon reducing this boundary-value problem to an 'initial value' problem for Korteweg-de Vries (K-dV) equations, reflection and transmission of a single soliton is investigated. In the following analysis, an effect of viscosity is perfectly neglected. Emphasis is placed on the effects of weak nonlinearity and dispersion how they suppress transmission for a high barrier. Finally we note that the present analysis can also be applied to the case of a vertically overhung watergate. In fact, the reduced boundary conditions take the same form as for the case of the barrier to the present order of approximation.

\section{Edge-Layer Theory}

\subsection{Summary of shallow-water theory}

Let us first summarize the main results of Boussinesq's theory for shallow-water waves. Taking the coordinates \(x\) and \(z\) in the two-dimensional configuration as shown in Fig. 1, the water-layer lying on the right- and the left-hand sides of the barrier located at \(x=0\) is referred to as region 1 and region 2. A subscript \(j(=1, 2)\) is attached to physical quantities to indicate to which region they belong. Whenever this subscript \(j\) is used without explicit reference \(j(=1, 2)\), it should be understood implicitly to take \(j=1\) and \(j=2\).

In each region, let a common depth of water-layer be \(h\), while let a characteristic wavelength and a characteristic elevation be, respectively, \(l\) and \(a\). Assuming inviscid and irrotational wave motions, Boussinesq's theory takes account of the weak nonlinearity and dispersion specified by the two small parameters \(\alpha\) and \(\beta\):

\begin{equation}
\alpha = a/h \ll 1, \text{ and } \beta = (h/l)^2 \ll 1,
\end{equation}

where \(\alpha\) and \(\beta\) are assumed to be of comparable order of magnitude. This implies that \(x\) and \(z\) are normalized by the different scales \(l\) and \(h\), respectively, while the time \(t\) is normalized by a characteristic time \(l/(gh)^{1/2}\), where \(g\) is the acceleration due to gravity.

The velocity potentials \(\phi_j(x, z, t)\) in each region are described by the respective Laplace equations together with the boundary conditions at the bottom \(z=-1\) and at the free surface \(z=\alpha \eta_j(x, t)\):

\begin{align}
\beta \phi_{j,zz} + \phi_{j,z} = 0 & \quad \text{for } -1 < z < \alpha \eta_j, \quad (2.2) \\
\phi_{j,z} = 0 & \quad \text{at } z = -1, \quad (2.3) \\
\eta_i + \phi_{j,z} + \frac{1}{2} \left( \alpha \phi_{j,x} + \frac{\alpha}{\beta} \phi_{j,z} \right) = 0 & \quad \text{at } z = \alpha \eta_j, \quad (2.4a) \\
\phi_{j,z} - \beta \eta_{j,z} - \alpha \beta \phi_{j,x} \eta_{j,z} = 0 & \quad (2.4b)
\end{align}

where \(\phi_j(x, z, t)\) and \(\eta_j(x, t)\) are normalized, respectively, by \(agl/(gh)^{1/2}\) and \(a\); a comma ',' in the subscripts designates a partial differentiation with respect to a variable(s) indicated after the comma. Here the boundary condition at the barrier is discarded on assuming such distant shallow-water regions that an effect of the barrier disappears.

On account of the small parameter \(\beta\), the Laplace equations (2.2) subject to the conditions (2.3) can be solved formally as

\begin{equation}
\phi_j = \sum_{n=0}^{\infty} (-\beta)^n \frac{(z+1)^n}{(2n)!} \frac{\partial^{2n} f_j}{\partial x^{2n}},
\end{equation}

where \(\phi_j(x, z, t)\) and \(\eta_j(x, t)\) are normalized, respectively, by \(agl/(gh)^{1/2}\) and \(a\); a comma ',' in the subscripts designates a partial differentiation with respect to a variable(s) indicated after the comma. Here the boundary condition at the barrier is discarded on assuming such distant shallow-water regions that an effect of the barrier disappears.
where $f_j$, the lowest order terms in the velocity potentials, depend on $x$ and $t$. Introducing (2.5) into (2.4) and retaining terms up to $O(\alpha, \beta)$, we have the following Boussinesq equations for $f_j$

$$f_{j,t} - f_{j,xx} - \frac{\beta}{3} f_{j,xxx} = -\alpha \left( \frac{1}{2} f_{j,t} + f_{j,x} \right)_t + O(\alpha^2, \alpha\beta, \beta^2),$$

and the respective elevations $\eta_j$ are given by

$$\eta_j = -\frac{f_{j,t}}{2} - \int f_{j,x} + \frac{\beta}{2} f_{j,xxx} + O(\alpha\beta, \beta^2).$$

### 2.2 Matched-asymptotic expansion method

Let us now consider the edge-layer in the vicinity of the barrier located at $x=0$. Since the edge-layer is confined horizontally in a narrow region comparable with the depth-scale but much smaller than the characteristic wavelength, the matched-asymptotic expansion method can be successfully applied. With the small parameter $\beta$, a new horizontal coordinate $\xi(=\beta^{-1/2}x)$, which implies the re-normalization of $x$ in terms of $h$, is introduced to magnify the narrow region around $x=0$. As $|\xi|$ becomes sufficiently large, the edge-layer naturally disappears and the shallow-water regions recover. According to the matched-asymptotic expansion method, this transition takes place at the so-called matching regions located at $\xi \to (-1)^{j+1} \infty$ but $|x| \to 0$. The velocity potentials and the elevations to be matched there are derived by expanding the shallow-water solutions for $\phi_j$ and $\eta_j$ in (2.5) and (2.7) around $x=0$ and setting $x = \beta^{1/2} \xi$:

$$\phi_{j(-1)^{j+1} \infty} = \left[ f_j + \beta^{1/2} f_{j,xx} \left[ \xi^2 - (z+1)^2 \right]/2 + \beta^{3/2} f_{j,xxx} \left[ \xi^3 - 3\xi (z+1)^2 \right]/6 \right]_{x=0} + O(\beta^2),$$

and

$$\eta_{j(-1)^{j+1} \infty} = \left\{ -\frac{f_{j,t}}{2} - \beta^{1/2} f_{j,xt} \xi - \frac{\alpha}{2} f_{j,x} + \frac{\beta}{2} f_{j,xxx} (1 - \xi^2) \right\}_{x=0} + O(\alpha\beta^{1/2}, \beta^3),$$

where note that $f_j$ and their derivatives are evaluated at $x=0$ so that they depend only on $t$. Hereafter in this section, we shall omit, for simplicity, the symbol $\{ \cdot \}_x=0$ indicating the evaluation at $x=0$, so that $f_j$ and their derivatives should be interpreted as those at $x=0$. Thus the matching conditions are given by

$$\tilde{\phi}_j \to \phi_{j(-1)^{j+1} \infty}$$

and

$$\tilde{\eta}_j \to \eta_{j(-1)^{j+1} \infty} \quad \text{as} \quad \xi \to (-1)^{j+1} \infty,$$

(2.10)

where $\tilde{\phi}_j(\xi, z, t)$ and $\tilde{\eta}_j(\xi, t)$ denote, respectively, the velocity potentials and the surface elevations in the edge-layer. Although the governing equations in the edge-layer are, of course, the Laplace equations subject to the boundary conditions at the bottom and the free surface, we must now take additional account of the boundary conditions along the barrier together with the matching conditions given above.

In view of the matching conditions (2.10), it is convenient to express the velocity potentials in the edge-layer in terms of the deviations $\psi_j(\xi, z, t)$ from $\phi_{j(-1)^{j+1} \infty}$:

$$\tilde{\phi}_j = \phi_{j(-1)^{j+1} \infty} + \beta^{1/2} \psi_j.$$  

(2.11)

Then, the matching conditions are simply given by

$$\psi_j \to 0 \quad \text{as} \quad \xi \to (-1)^{j+1} \infty.$$  

(2.12)

Introducing (2.11) into (2.2)-(2.4) after effecting the change of variables from $x$ to $\xi$, and expanding the boundary conditions at the free surface around the quiescent level $z=0$, we have

$$\psi_{j,\xi \xi} + \psi_{j,zz} = 0 \quad \text{for} \quad -1 < z < 0,$$

$$\psi_{j,z} = 0 \quad \text{at} \quad z = -1,$$

$$\psi_{j,z} = \alpha f_{j,\xi} \psi_{j,zz} - \beta \psi_{j,zt} + O(\alpha \beta^{1/2}, \beta^{3/2}),$$

$$\tilde{\eta}_j = \eta_{j(-1)^{j+1} \infty} - \beta^{1/2} \psi_{j,t} - \alpha \left( \psi_{j,\xi} + 2 f_{j,\xi} \psi_{j,\xi} + \psi_{j,\xi} \right) / 2 + O(\alpha \beta^{1/2}, \beta^{3/2})$$

at $z=0$.  

(2.13a)  

(2.14)  

(2.15a)  

(2.15b)
We now consider the boundary conditions at $\xi = 0$. With $\psi_j$ thus introduced, they take a little complicated from. Along the barrier, the fluid motions should be tangential, i.e., $\phi_{j,\xi} = 0$. In terms of $\psi_j$, this becomes

$$\psi_{j,\xi} = -f_{j,x} + \beta f_{j,xxx}(z + 1)^2/2 + O(\beta^{3/2}) \quad \text{at} \quad \xi = 0 \quad \text{and} \quad -1 < z < -r,$$

(2.16)

where $r$ is a ratio of the opening (the ‘fictitious’ cut shown by the dotted line in Fig. 1) to the total still-water depth. Along the cut, the potentials ($\tilde{\phi}_j$ and the horizontal velocities) $\tilde{\phi}_{j,\xi}$ must be continuous in order to connect both regions 1 and 2 without any physical discontinuity (note that the potentials may have a jump across the cut by an amount of arbitrary absolute constant $C$). If the potentials are thus connected, it will turn out that the surface elevations are automatically connected continuously at $\xi = 0$. Thus the boundary conditions along the cut are given by $\tilde{\phi}_1 - \tilde{\phi}_2 = C$ and $\tilde{\phi}_{1,\xi} - \tilde{\phi}_{2,\xi} = 0$. When these conditions are expressed in terms of $\psi_j$, we have

$$\psi_1 - \psi_2 = -\beta^{-1/2}(f_1 - f_2 - C) + \beta^{1/2}(f_{1,xxx} - f_{2,xxx})(z + 1)^2/2 + O(\beta^{3/2})$$

at $\xi = 0$ and $-r < z < 0$, (2.17a)

and

$$\psi_{1,\xi} - \psi_{2,\xi} = -(f_{1,x} - f_{2,x}) + \beta(f_{1,xxx} - f_{2,xxx})(z + 1)^2/2 + O(\beta^{3/2})$$

at $\xi = 0$ and $-r < z < 0$. (2.17b)

These boundary conditions together with (2.14) and (2.15a) constitute the boundary-value problems for the Laplace equations (2.13) in each semi-infinite rectangular region. Since the boundary conditions except (2.17a) are given in a form of the normal derivatives, the boundary-value problem may be regarded as a Neumann’s problem, provided $\psi_{j,\xi}$ were known along $\xi = 0$. Then, compatibility conditions are required for the Neumann’s problem to be solved consistently. In the present context, the conditions correspond to the volume conservation in each edge-layer. Invoking Stoke’s theorem:

$$\oint_{\partial S_j} (\psi_{j,\xi} \, dx - \psi_{j,\xi} \, dz) = 0,$$

(2.18)

where $\partial S_j$ denotes the boundary of the region $j$ including the cut along $\xi = 0$ and the integration is carried out in the positive sense (counterclockwise) along $\partial S_j$, it follows that

$$\int_{-1}^{0} \psi_{1,\xi}(0, z, t) \, dz = \int_{0}^{\infty} (\alpha f_{1,\xi} \psi_{1,\xi} - \beta \psi_{1,\zeta}) z = 0 \, d\xi + O(\alpha \beta^{1/2}, \beta^{3/2}),$$

(2.19a)

and

$$\int_{-1}^{0} \psi_{2,\xi}(0, z, t) \, dz = \int_{0}^{\infty} (\beta f_{2,\xi} \psi_{2,\xi} + \alpha \psi_{2,\zeta}) z = 0 \, d\xi + O(\alpha \beta^{1/2}, \beta^{3/2}).$$

(2.19b)

Elimination of $\psi_{j,\xi}(0, z, t)$ by using (2.16) and (2.17b) yields

$$-f_{1,x} + f_{2,x} = \alpha \left( f_{1,\xi} \int_{0}^{\infty} \psi_{1,\xi} \, d\xi + f_{2,\xi} \int_{-\infty}^{0} \psi_{2,\xi} \, d\xi \right)_{z = 0}$$

$$-\frac{\beta}{6} (f_{1,xxx} - f_{2,xxx}) - \beta \left( \int_{0}^{\infty} \psi_{1,\zeta} \, d\xi + \int_{-\infty}^{0} \psi_{2,\zeta} \, d\xi \right)_{z = 0} + O(\alpha \beta^{1/2}, \beta^{3/2}).$$

(2.20)

Since the derivatives of $f_j$ have been evaluated at $x = 0$, (2.20) provides a relation between the boundary-values in the two shallow-water regions separated by the edge-layer and therefore plays a role of a boundary condition to Boussinesq equations (2.6). In this sense, (2.20) is called a ‘reduced’ boundary condition which reflects an effect of inner structure of the edge-layer on the shallow-water regions. If we neglect the right-hand side of (2.20), it reduces to one of Lamb’s con-
dictions representing the continuity of volume flux. It should be noted, however, that the correction terms of \(O(\alpha, \beta)\) will turn out to vanish identically so that one of Lamb’s conditions is still valid within the present order of approximation. To evaluate (2.20) up to \(O(\alpha, \beta)\) inclusive, it is sufficient to specify \(\psi_j\) within the lowest order. In connection with the other reduced boundary condition, however, we shall next show a procedure to seek \(\psi_j\) up to \(O(\beta^{1/2})\).

### 2.3 Edge-layer solutions

As far as \(\psi_j\) are sought up to \(O(\beta^{1/2})\), the right-hand side of the boundary conditions (2.15a) may be neglected, i.e., the free surface may be regarded as if it were a rigid wall. If the boundary values \(\psi_j(0, z, t) = g_j\) were specified, the solutions \(\psi_j\) in the semi-infinite rectangular regions are obtained as

\[
\psi_j = (-1)^{j+1} \int_0^0 G_j(z', \xi, z) g_j(z') \, dz',
\]

(2.21a)

with

\[
G_j(z'; \xi, z) = \frac{1}{2\pi} \log \left| \cos (\pi z') - \cosh (\pi \xi) \cos (\pi z) \right|^2 + \sinh^2(\pi \xi) \sin^2(\pi z),
\]

(2.21b)

where \(G_1\) and \(G_2\) denote the Green’s functions in each semi-infinite rectangular region.\(^3\)\(^4\)

Substituting (2.21) into (2.17a), the problem is recast into solving an integral equation for the unknowns \(g_j\) under the condition (2.17b). It is now convenient to work with the physical horizontal velocities \(u_j\) rather than \(g_j\):

\[
u_j = \beta^{-1/2} G_j, \quad f_j, u_j + g_j + O(\beta).
\]

(2.22)

In terms of \(u_j\), the boundary conditions at \(\xi = 0\) take simply

\[
\begin{align*}
  u_1 &= u_2 = 0 \quad \text{for} \quad -1 < z < -r, \\
  u_1 &= u_2 = u \quad \text{for} \quad -r < z < 0,
\end{align*}
\]

(2.23)

and the compatibility conditions (2.19) are rewritten as

\[
\int_{-r}^0 u(z) \, dz = f_{1,xx} = f_{2,xx},
\]

(2.24)

where we have omitted higher order terms than \(\beta^{1/2}\). Using \(\psi_j\) in (2.21) with \(g_j = u_j - f_{j,xx}\), one obtains the following integral equation for \(u(z)\) from the continuity of the velocity potentials (2.17a):

\[
\frac{2}{\pi} \int_{-r}^0 \log | \cos (\pi z') - \cos (\pi z) | u(z') \, dz' = p_0 + \beta^{1/2} p_2, \quad -r < z < 0,
\]

(2.25a)

where

\[
p_0 = -\beta^{-1/2} (f_1 - f_2 - C) - (\log 4) f_{1,xx} / \pi + \beta^{1/2} (f_{1,xx} - f_{2,xx}) / 2,
\]

and

\[
p_1 = 2p_2 = f_{1,xx} - f_{2,xx}.
\]

Upon expressing a solution \(u(z)\) in the form of

\[
u(z) = p_0 u^{(0)} + \beta^{1/2} (p_1 u^{(1)} + p_2 u^{(2)}),
\]

(2.26)

where \(u^{(k)}(z) (k = 0, 1, 2)\) satisfy the equations

\[
\frac{2}{\pi} \int_{-r}^0 \log | \cos (\pi z') - \cos (\pi z) | u^{(k)}(z') \, dz' = z^k, \quad -r < z < 0,
\]

(2.27)

and substituting (2.26) into (2.24), we can obtain the other ‘reduced’ boundary condition:

\[
f_1 - f_2 - C = \beta^{1/2} \kappa_1 f_{1,xx} + \beta \kappa_2 (f_{1,xx} - f_{2,xx}) + O(\alpha \beta^{1/2}, \beta^{3/2}),
\]

(2.28a)

where

\[
\kappa_1 = -[(\log 4) / \pi + 1/s^{(0)}], \quad \kappa_2 = 1/2 + s^{(1)} / s^{(0)} + s^{(2)} / (2s^{(0)}),
\]

(2.28b)

and

\[
s^{(k)} = \int_{-r}^0 u^{(k)}(z) \, dz.
\]

Differentiating (2.28a) with respect to \(t\) for eliminating the constant \(C\), one obtains a generalized version of the other Lamb’s condi-
tion representing the continuity of surface elevation. Indeed, this condition, having been derived from the continuity of potentials, will be found to yield the continuity of elevations.

To obtain the explicit expression of \( \kappa_1 \), the integral equation (2.27) must be solved. Fortunately, it can be solved analytically after being reduced to a singular integral equation of Cauchy's type.\(^7,8\) Although (2.27) is solvable for \( k = 0, 1, 2 \), only \( u^{(0)} \) is necessary to evaluate \( \kappa_1 \). Therefore the explicit expressions of \( u^{(1)} \) and \( u^{(2)} \) are not given here. Changing variables in (2.27) from \( z' \) to \( w' \) via

\[
\cos (\pi z') = \frac{1}{2} \left( 1 + J \right) + \frac{1}{2} \left( 1 - J \right) w',
\]

\[
J = \cos (\pi r), \quad (2.29)
\]

and similarly from \( z \) to \( w \), and differentiating the resultant with respect to \( w \), it follows that

\[
P \int_{-1}^{1} \frac{1}{w' - w} u[z'(w')] \frac{u[z'(w')]}{\sqrt{1 - \frac{1}{4} (1 + J + (1 - J)w')^2}} dw' = 0, \quad (2.30)
\]

where \( P \) implies Cauchy's principal value. The solution to this integral equation is obtained as\(^7,8\)

\[
\frac{u[z'(w')]}{\sqrt{1 - \frac{1}{4} (1 + J + (1 - J)w')^2}} = -\frac{c_0}{\sqrt{1 - w'^2}}, \quad (2.31)
\]

where \( c_0 \) is an arbitrary constant to be determined for (2.31) to satisfy the original equation (2.27) with \( k = 0 \). As this consequence, we have

\[
c_0 = \frac{\pi}{4K \log K}, \quad \text{with} \quad K = \frac{1}{2} \sin^2 \left( \frac{\pi r}{2} \right). \quad (2.32)
\]

Thus \( u^{(0)}(z) \) is obtained as

\[
u^{(0)}(z) = \frac{\pi}{2 \log K} \sqrt{\cos (\pi z) + 1 \cos (\pi z) - \cos (\pi r)}, \quad (2.33)
\]

so that \( s^{(0)} \) and \( \kappa_1 \) can be evaluated as follows:

\[
s^{(0)} = \frac{\pi}{2 \log K} \quad \text{and} \quad \kappa_1 = -\frac{2}{\pi} \log (2K) = -\frac{4}{\pi} \log \left[ \sin \left( \frac{\pi r}{2} \right) \right]. \quad (2.34)
\]

We remark here on the accuracy of the two reduced boundary conditions (2.20) and (2.28a). Using (2.28a) in (2.20) and the anti-symmetry of \( \psi_i \), i.e., \( \psi_i(\xi, 0, t) = -\psi_i(-\xi, 0, t) \) to be proven later, it is found that the correction terms involving \( \psi_j \) drop out. In addition, evaluating Boussinesq equations (2.6) at \( x = 0 \), it is found that \( f_{1,xx} - f_{2,xx} = (f_{1,x} - f_{2,x})_{t} + O(\alpha, \beta) = O(\alpha, \beta) \) because of the lowest order relation of (2.20). Hence all the correction terms of \( O(\alpha, \beta) \) in (2.20) vanish so that Lamb's condition for the continuity of volume flux is still valid up to \( O(\alpha, \beta) \) inclusive. But this is not the case for the other boundary condition representing the continuity of elevation. As far as \( \kappa_1 \) remains of \( O(1) \), the correction term is of \( O(\beta^{1/2}) \). But it should be noted that \( \kappa_1 \) diverges as \( r \) tends to zero. Thus the correction term \( \beta^{1/2} \kappa_1 f_{1,xx} \) resulting from the first term on the right-hand side of (2.28a) becomes significant, whereas \( \kappa_2 \) remains finite even in this limit because \( s^{(1)}/s^{(0)} \) and \( s^{(2)}/s^{(0)} \) behave, respectively, as \( -r/2 \) and \( r^2/3 \) as \( r \to 0 \) (which can be verified by integrating (2.27) with respect to \( z \) over the region \(-r < z < 0\)). Even for such a case with \( \kappa_1 \sim O(\beta^{-1/2}) \), the reduced boundary condition (2.28a) still holds with the error of \( O(\beta) \). In passing, we remark that these reduced boundary conditions can also be applied to the case of a vertically overhung watergate with an opening underneath, i.e., the gate along \(-r < z < r\) and the opening along \(-1 < z < r\). As far as (2.28) is taken up to \( O(\beta^{1/2}) \), \( r \) has only to be replaced by \( 1 - r \).

In concluding this section, we now calculate the surface elevations in the edge-layer by using (2.15b). To do so, the values of \( \psi_j \) at \( z = 0 \) are necessary. From (2.21) together with (2.22) and (2.23), they are expressed as

\[
\psi_j(\xi, 0, t) = \frac{1}{\pi} \left\{ (\log 2) f_{j,x} + (-1)^{j+1} \left[ -\pi f_{j,x} \xi + \int_{-r}^{0} \log |\cos (\pi z') - \cosh (\pi \xi)| u(z') \, dz' \right] \right\}, \quad (2.35)
\]
where the following integral formula has been used:

\[
\int_0^\pi \log (a + b \cos x) \, dx = \pi \log \left[ \frac{1}{2} \left( a + \sqrt{a^2 - b^2} \right) \right]
\]

\[= 2\pi \log \left[ \frac{1}{2} \left( \sqrt{a-b} + \sqrt{a+b} \right) \right], \quad (a \geq |b|).
\] (2.36)

Substituting \( u(z') = p_0 u^{(0)} + O(\beta^{1/2}) = f_{1,x} u^{(0)}/z^{(0)} + O(\beta^{1/2}) \) and carrying out the integration, it leads to

\[
\psi_j(\xi, 0, t) = (-1)^{j+1} \frac{2}{\pi} f_{j,x} \log \left\{ \frac{1}{2} \left[ 1 - \exp (-\pi |\xi|) \right] \right\}
\]

\[+ \frac{1}{2} \sqrt{1-2 \cos (\pi r) \exp (-\pi |\xi|) + \exp (-2\pi |\xi|)} \}, \quad (2.37)
\]

where the transformation (2.29) from \( z' \) to \( w' \) and the formula (2.36) have been employed. We have thus confirmed the antisymmetry of \( \psi_i, \) i.e., \( \psi_i(\xi, 0, t) = -\psi_i(-\xi, 0, t) \).

Using (2.37) in (2.15), the \( \xi \)-dependence of the surface elevations in the edge-layer can be specified up to \( O(\beta^{1/2}) \). Of course, explicit surface elevations can be determined only after the outer shallow-water solutions \( f_j \) were obtained. At present, it is interesting to examine the surface elevations at \( \xi = 0 \). They are immediately obtained as

\[
\eta_i = -f_{1,x} + \beta^{1/2} \frac{K_1}{2} f_{1,x},
\]

(2.38)

and

\[
\eta_2 = -f_{2,x} - \beta^{1/2} \frac{K_1}{2} f_{1,x},
\]

from which it is indeed verified that the surface elevations are continuous across \( \xi = 0 \), provided the reduced boundary condition (2.28) holds.

§3. Reflection and Transmission of a Soliton

Let us now investigate reflection and transmission of a single soliton incident upon a barrier. Since Boussinesq equations (2.6) are used, it may be appropriate here to remark on the accuracy of approximation involved. In light of the form of Boussinesq equation, one might think that it could specify \( f \) up to \( O(\alpha, \beta) \) inclusive. In fact, however, it cannot give a correct evaluation of \( f \) up to \( O(\alpha, \beta) \). If a specification up to this order would be desired, the higher-order version than Boussinesq’s theory must be employed. In consistent with the approximation involved in Boussinesq’s theory, it is sufficient here to take account of the reduced boundary conditions up to \( O(\beta^{1/2}) \):

\[
f_{1,x} - f_{3,x} = 0,
\]

\[
f_{1,t} - f_{2,t} = \beta^{1/2} K_1 f_{1,xt}, \quad \text{at} \quad z = 0.
\]

(3.1a)

(3.1b)

To seek solutions to Boussinesq equations, \( f \) are assumed to be expressed approximately by a superposition of the right- and left-going waves:

\[
f_j = F_j^+(\sigma^+, \tau) + F_j^-(\sigma^-, \tau) + O(\alpha, \beta),
\]

(3.2)

with

\[
\sigma^+ = t-x, \quad \sigma^- = t+x \quad \text{and} \quad \tau = \alpha x,
\]

where \( \sigma^+ \) and \( \sigma^- \) denote the time measured, respectively, in the moving frames with the linear velocity to the positive and the negative directions of the \( x \)-axis, and \( \tau \) denotes the ‘long-space’ scale. Introducing (3.2) into (2.6), and retaining the lowest order terms, the surface elevations \( \eta_j^\pm (\equiv -F_j^\pm; \text{the alternate sign vertically ordered}) \) are described, respectively, by K-dV equations of the following form:

\[
\frac{\partial \eta_j^+}{\partial \tau} + \frac{3}{2 \eta_j^+} \frac{\partial \eta_j^+}{\partial \sigma^+} + \frac{\beta}{6 \alpha} \frac{\partial^3 \eta_j^+}{\partial \sigma^+^3} = 0. \]

(3.3)

Thus solving Boussinesq equations is now reduced to solving each K-dV equation for the incident, reflected and transmitted waves.

Let us consider a soliton incident upon the barrier from the region 1. In this case, \( f_1 \) consists of the incident wave \( F_1^- \) and the reflected wave \( F_1^+ \), whereas \( f_2 \) takes the transmitted
wave $F_\tau^2$ only. In terms of the surface elevations, they are expressed as

$$
\eta_1 = Y_I(\sigma^-, \tau) + Y_R(\sigma^+, \tau), \quad (3.4a)
$$
$$
\eta_2 = Y_T(\sigma^-, \tau), \quad (3.4b)
$$
where the suffixes $I$, $R$, and $T$ denote, respectively, the incident, reflected, and transmitted waves. As the incident wave $Y_I$, we take the following soliton-solution with unit peak at $\sigma^- = c^-$ and $\tau = 0$:

$$
Y_I = \text{sech}^2 \left[ D \left( \sigma^+ - \frac{\tau}{2} - c^- \right) \right],
$$
with

$$
D = \sqrt{\frac{3\alpha}{4\beta}}, \quad (3.5)
$$
whereas $Y_R$ and $Y_T$ are unknown at this stage.

Imposing the reduced boundary conditions (3.1) on (3.4), we have

$$
\begin{align*}
\mu \dot{Y}_T + Y_T &= Y_I, \quad (3.6a) \\
Y_R + Y_T &= Y_I, \quad (3.6b)
\end{align*}
$$
with

$$
\mu = \beta^{1/2} \kappa_1 / 2, \quad (3.6c)
$$
where $\sigma^+ = \sigma^- = t$ at $\tau = 0$ and the dot designates the differentiation with respect to $t$. As far as $\mu$ is of $O(\beta^{1/2})$, the term with the dot in (3.6a) can be incorporated, by Taylor’s theorem, to the phase of $Y_T$:

$$
\mu \dot{Y}_T + Y_T = Y_T(t + \mu, 0) + O(\beta). \quad (3.7)
$$
Thus $Y_T$ has the same form as that of the incident soliton with the small phase shift $\mu$:

$$
Y_T(t, 0) = Y_I(t - \mu, 0), \quad (3.8a)
$$
while $Y_R$ is obtained simply from (3.6b) as

$$
Y_R(t, 0) = Y_I(t, 0) - Y_T(t - \mu, 0)
= \mu \dot{Y}_I(t, 0) + O(\beta). \quad (3.8b)
$$
These boundary-values (3.8) serve as the ‘initial’ conditions for K-dV equations (3.3) at $\tau = 0$. Therefore an evolution of the reflected and transmitted waves is determined by solving the ‘initial’ value problem. Here we note that the evolution concerned here is spatial but not temporal in the usual sense.

Since the transmitted wave (3.8a) is exactly a soliton-solution, it obviously continues to propagate as a soliton. Thus, for a moderate height of the barrier, the incident soliton can pass over it as if the barrier were transparent, but the effect of the barrier is memorized in the form of a phase shift. For the reflected wave, on the other hand, $Y_R$ is a small quantity of $O(\beta^{1/2})$. If this order is neglected, we recover Lamb’s results of no reflection.

For a relatively high barrier for which $\mu = O(1)$, the above argument becomes incorrect. In this case, (3.6a) should be regarded as a differential equation for $Y_T$. Defining $\chi = D\tau$, it can be written as

$$
q \frac{dY_T}{d\chi} + Y_T = \text{sech}^2 \chi, \quad (3.9a)
$$
where

$$
q = \mu D = \frac{\sqrt{3\alpha \kappa_1}}{4} = -\frac{3\alpha}{\pi} \log \left( \sin \left( \frac{\pi r}{2} \right) \right). \quad (3.9b)
$$
Here note that $Y_T$ is solely determined by the parameter $q$ which is a combination of $r$ and $\alpha$, but independent of $\beta$. Solving (3.9), $Y_T$ and therefore $Y_R$ (by (3.6b)) can be obtained analytically. But they are too complicated to be set on the scheme of the inverse scattering method to investigate an evolution of the reflected and transmitted waves. Therefore K-dV equations are solved numerically by the well-known explicit finite difference method.\(^{10}\) To do so, the ratio $\alpha / \beta$, known as Ursell’s parameter $U_r$, must be given. Taking $U_r = 9.25$ (the reason for this choice, see ref. 9), several numerical computations are carried out. Taking a typical case with $r = 0.05$, $\alpha = 0.2$ and $c^- = 0$ in (3.5), we show in Fig. 2 the bound-

![Fig. 2](image-url)
ary-values (to be used as the ‘initial’ values for K-dV equations) for the reflected and transmitted waves obtained by solving (3.9) and (3.6b). It is seen that the transmitted wave is reduced in amplitude and is slightly lengthened in width, while the reflected wave is enhanced and also lengthened with the positive and negative areas cancelling each other. These properties can also be understood from the following conditions derived directly from (3.6a) and (3.9):

$$\int_{-\infty}^{\infty} Y_T(t, 0) \, dt = \int_{-\infty}^{\infty} Y_r(t, 0) \, dt > 0, \quad (3.10a)$$

and

$$\int_{-\infty}^{\infty} Y_R(t, 0) \, dt = 0. \quad (3.10b)$$

Since the initial condition (3.10a) has the positive area, the transmitted wave evolves necessarily into, at least, one soliton. From the reflected wave, on the other hand, the part with positive value will evolve into a soliton(s) whereas the part with negative value will evolve into slowly decaying ripples. In Figs. 3 and 4, we show, respectively, the spatial evolution of the reflected and transmitted waves from the initial conditions depicted in Fig. 2. It is seen that the transmitted wave reshapes into a soliton-form, while the reflected wave evolves into one small soliton together with decaying ripples.

Let us finally consider reflection and transmission from a viewpoint of energy balance. Noting that the potential and kinetic energies are equal to each other within the present approximation, it is found, by effecting a Fourier transform on (3.6), that the energy partitioned into the reflected and transmitted waves must satisfy the following relation:

$$E_R + E_T = E_i, \quad \text{at} \quad \tau = 0, \quad (3.11a)$$

where

$$E_R = \int_{-\infty}^{\infty} |Y_R|^2 \, d\chi = \int_{-\infty}^{\infty} \frac{(qk)^2}{1 + (qk)^2} F(k) \, dk, \quad E_T = \int_{-\infty}^{\infty} |Y_T|^2 \, d\chi = \int_{-\infty}^{\infty} \frac{1}{1 + (qk)^2} F(k) \, dk,$$

and

$$E_i = \int_{-\infty}^{\infty} |Y_i|^2 \, d\chi = \int_{-\infty}^{\infty} F(k) \, dk = \frac{4}{3}, \quad F(k) = \frac{\pi}{2} k^2 \cosh^2 \left( \frac{\pi k}{2} \right). \quad (3.11b)$$

![Fig. 3. Spatial evolution of the reflected wave.](image-url)
Since the energy is a conserved quantity of KdV equation, each energy partitioned at $\tau = 0$ must be conserved for the reflected and transmitted waves. The ratio of the energy shared by the reflected and transmitted waves to the incident wave can be evaluated by (3.11). In Fig. 5, the ratio of the energy shared by the reflected wave to that of the incident one, i.e., $E_R/E_i = R$, is shown as a function of $r$ for three typical values of $\alpha$. For a moderate height of the barrier, it is seen that most of the energy of the incident wave is carried away by the transmitted wave. As the barrier becomes higher, however, the energy of the reflected wave becomes significant and transmission is suppressed considerably for an extremely high barrier. For such a narrow opening, however, a role of viscosity may come into play but this problem still remains as an open question.

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References