EVOLUTION OF NONLINEAR ACOUSTIC WAVES IN A GAS-FILLED PIPE

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ABSTRACT

Boundary-value problems are solved for the spatial evolution equation which describes the nonlinear acoustic waves propagating unidirectionally in a long gas-filled pipe. Neglecting the dissipation due to the diffusivity of sounds, concerned is the effect of wall friction incorporated in the form of a hereditary integral known as the fractional derivative of order 1/2. The wall friction fails to balance with the nonlinearity to allow an emergence of a discontinuity eventually. The typical evolution from the step function and the pulse shows that the fractional derivative gives rise to the global effect in contrast with the local one by the second-order derivative due to the diffusivity of sounds, active only where a waveform is steep enough.

INTRODUCTION

When an acoustic shock wave is propagated in a long gas-filled pipe, it is subjected to wall friction. The friction is caused by a boundary layer produced adjacent to the pipe wall. This kind of dissipation emerges differently from the one due to the diffusivity of sounds in that it accumulates, as the wave is propagated down the pipe, to give a hereditary effect. Even for the memoryless Newtonian fluids, the boundary layer can yield the hereditary effect. In fact, the shear stress at the wall is given by an integral of the velocity outside of the boundary layer with the kernel of algebraically decreasing function of time [1-3]. To the well-known nonlinear hyperbolic evolution equation, the wall friction introduces the above type of hereditary integral so-called the fractional derivative of order 1/2 [3].

This paper concerns how the hereditary effect affects the unidirectional propagation of nonlinear acoustic waves. On the basis of the spatial evolution equation, a boundary-value problem is solved under such typical conditions as the step function and the Gaussian shaped pulse. The second-order derivative due to the diffusivity of sounds is neglected because its effect is so small compared with the wall friction that it comes into play very locally to check the nonlinear steepening in waveform. In contrast, it is shown that the wall friction appears globally in a sense that its effect prevails over a whole waveform. In application, such a knowledge provides a theoretical basis in dealing with acoustic problems in a tunnel generated by future high-speed trains.
EQUATION EQUATION

We first present the spatial evolution equation for nonlinear acoustic waves propagating unidirectionally in a long pipe of uniform cross-section. Effect of viscosity is assumed to be so small that the thickness of a boundary layer is negligible compared with a radial scale of the pipe. Outside of the boundary layer, the axial motion of gas is uniform over the cross-section. Evolution equation is then given by (for its derivation, see [1-3]):

\[
\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial \theta} = - \delta \frac{\partial^{1/2} f}{\partial \theta^{1/2}}, \tag{1}
\]

with the definition of the fractional derivative of order 1/2 [3]:

\[
\frac{\partial^{1/2} f}{\partial \theta^{1/2}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\theta} \frac{1}{\sqrt{\theta - \theta'}} \frac{\partial f}{\partial \theta'} d\theta', \tag{2}
\]

where \( f \) denotes the axial-velocity component \( u \) of gas through the definition \( ((\gamma + 1)/2)u/a = \varepsilon f \), \( a \) being the sound speed and \( \gamma \) and \( \varepsilon \) specifying, respectively, the ratio of the two specific heats and the order of nonlinearity. Here \( \theta = \omega(t-x/a) \) and \( \xi = \varepsilon \omega x/a \) denote, respectively, the retarded time measured in a frame moving with the sound speed and the long space-coordinate associated with the order of nonlinearity, \( x \) and \( r \) being the space-coordinate and the time, respectively and \( \omega \) a characteristic frequency. The effect of wall friction appears through the constant \( \delta \) \(=[C(\nu/\omega)^{1/2}/(\varepsilon R)] \) where \( \nu \) and \( R \) are the kinematic viscosity of gas and a characteristic radius of the pipe, and \( C \) is a constant given by \( C=1+(\gamma-1)/Pr^{1/2}, \) \( Pr \) being the Prandtl number. Finally we estimate the smallness of the second-order derivative neglected in (1). Its coefficient \( \mu \) is given by \( \nu_d \omega^2/(2\varepsilon a^2) \) where \( \nu_d \) is the diffusivity of sounds [3]. For acoustic waves generated in a tunnel, for example, \( \delta \) takes \( 5.6 \times 10^{-4}/\varepsilon \) for \( \omega=2\pi \) rad/s and \( R=4m \), whereas \( \mu \) takes \( 1.1 \times 10^{-9}/\varepsilon \). The ratio of \( \mu/\delta \) is, indeed, of order of \( 10^{-6}! \)

BOUNDARY-VALUE PROBLEMS

We now consider an evolution of \( f \) by (1) subjected to a condition

\[
f(\theta, X=0) = F(\theta). \tag{3}
\]

Here note that this condition prescribes physically the boundary value for \( f, \) although (3) is mathematically an initial condition for (1). In order to solve (1) under (3), it is convenient, by analogy with the hyperbolic equation, to rewrite (1) formally in the 'characteristic form':

\[
\frac{df}{dX} = - \delta \frac{\partial^{1/2} f}{\partial \theta^{1/2}}, \tag{4.a}
\]

along the 'characteristics' defined by

\[
\frac{d\theta}{dX} = - f. \tag{4.b}
\]
Relations for a discontinuity

By the nonlinearity, \( f \) may become multi-valued in the course of evolution, although a role of the fractional derivative is unknown at this stage. Before proceeding to a specific boundary-value problem, therefore, we begin with examining whether or not (1) allows a propagation of a discontinuity, i.e., a shock wave.

Suppose that a discontinuity in \( f \) be at \( \theta = \tau(X) \) and that a continuous solution \( B(\theta, X) \) be prevalent before it, i.e., \( \theta < \tau(X) \). With this discontinuity inclusive, let a solution \( f \) be expanded in the following form:

\[
f = B(\theta, X) + [V_0 + V_1 \eta^{1/2} + V_2 \eta + \ldots + V_n \eta^{n/2} + \ldots] h(\eta),
\]

where \( \eta = \theta - \tau(X) \) and \( h(\eta) \) is the step function; \( V_n \) (\( n=0,1,2,\ldots \)) are functions of \( X \) and \( V_0 (\neq 0) \) gives a strength of discontinuity. We assume that \( B \), which satisfies, of course, (1) in \( \theta < \tau(X) \), is appropriately expanded around \( \theta = \tau(X) \) so that it can be continued beyond \( \theta = \tau(X) \) as follows:

\[
B = B_0 + B_2 \eta + \ldots + B_{2n} \eta^n + \ldots,
\]

where \( B_{2n} \) (\( n=0,1,2,\ldots \)) are functions of \( X \). In passing, the half powers of \( \eta \) in (5) are suggested in view of the formula of the fractional derivative [3]:

\[
\frac{d^{1/2}}{d\eta^{1/2}} \left| \frac{\eta^{p-1} h(\eta)}{\Gamma(p)} \right| \frac{\eta^{p-3/2} h(\eta)}{\Gamma(p-1/2)},
\]

where \( p \geq 1 \) and \( \Gamma(p) \) denotes the Gamma function.

Upon introducing (5) with (6) into (1) and noting that \( f \partial f / \partial \theta = 1/2, \partial^2 f / \partial \theta^2 \) and \( h(\eta)^2 = h(\eta) \), we have from the coefficients of the delta function \( \delta(\eta) \) and the functions \( |\eta|^{n/2} h(\eta) \) (\( n=1,2,3,\ldots \)), respectively:

\[
\frac{dx}{dx} = -(B_0 + \frac{V_0}{2}), \quad V_1 = \frac{4\delta}{\sqrt{\pi}}, \quad \ldots
\]

\[
V_n = \frac{4}{nV_0} \left( \frac{dV_{n-2}}{dx} - \frac{n}{4} (V_1 V_{n-1} + V_2 V_{n-2} + \ldots + V_{n-1} V_1)ight) \quad \ldots
\]

\[
+ \frac{\Gamma(n+1)}{\Gamma(n/2)} \frac{n!}{2} V_{n-1} - \frac{n}{2} (B_2 V_{n-2} + B_4 V_{n-4} + \ldots + B_n V_0), \quad (n = 2,3,4,\ldots)
\]

where \( B_{2n} \) vanish for odd \( n \). The relation (8.a) determines the velocity of the discontinuity. It is derived from the balance on the left-hand side of (1) only. In this respect, (1) still preserves the same local property as the hyperbolic equation. This property might be anticipated in advance because the order 1/2 of the fractional derivative is indeed lower in (1) than the highest derivative! Effect of the fractional derivative first appears in \( V_1 \). Interestingly, it is an absolute constant independent of the discontinuity in so far as \( V_0 \) does not vanish. Owing to this term, the discontinuous profile looks rounded. Hence it is found that (1) may allow the propagation of a
discontinuity, which cannot be checked by the fractional derivative unlike the second-order derivative in Burgers equation.

Typical examples
We now look at specific evolution from some typical boundary values. In the followings, four types of conditions are considered: the step functions and the Gaussian shaped pulses with the positive and the negative polarity, respectively. Evolution is calculated by integrating (4) numerically along each 'characteristic'. As soon as a waveform becomes multi-valued, the discontinuous solution (5) is fitted locally into the solution (although the relations (5), (6) and (8) assume $V_0$ of $O(1)$).

Positive step function: $F(\theta)=h(\theta)$

Because no characteristic scale of $\theta$ is involved in this condition, $\delta$ can be set equal to unity, without any loss of generality, by re-defining $\theta$ and $X$ in terms of $\theta/\delta^2$ and $X/\delta^2$, respectively. Figure 1 shows the spatial evolution of $f$ up to $X=0.7$ by the $X$-steps 0.05. Here the fore-region $\theta<\tau(X)$ is an undisturbed state with $f=\delta=0$.

Figure 2 shows the arrival time $\tau$ and the strength $V_0$ of the discontinuity with respect to the location $X$. In light of the fact that, without the wall friction, the shock wave advances with a constant speed $d\tau/dX=1/2$, Fig. 2 shows quantitatively how rapidly the shock wave is retarded by the friction. When $V_0$ vanishes at some location $X$, the discontinuity of infinitesimally small strength will diffuse but backward only so that the fore-region is kept intact, i.e., $f=0$ [4].

![Figure 1. Evolution of the positive step up to $X=0.7$ by the $X$-steps 0.05 for $\delta=1$.](image1)

![Figure 2. Spatial variations of $\tau(X)$ and $V_0(X)$ for the evolution in Fig. 1.](image2)

Negative step function: $F(\theta)=-h(\theta)$

In this case as well, $\delta$ can be set equal to unity. But instead of a discontinuous solution, a similarity solution is to appear in the sector region $0<\theta<X$ around the origin. Defining the similarity-variables by $\theta/X^2=\zeta$ and $fX=g(\zeta)$, $g$ is governed by

$$g - 2\zeta \frac{dg}{d\zeta} - g \frac{d^2g}{d\zeta^2} = -\frac{1}{\sqrt{\pi}} \int_0^\zeta \frac{1}{\sqrt{\zeta-\zeta'}} \frac{dg}{d\zeta'} d\zeta', \quad (9)$$

...
where the region in $\theta<0$ is undisturbed, i.e., $g=0$ so that the lower limit of the integration is truncated at $\zeta=0$. Analytical solution of (9) is difficult, but some asymptotic solutions are easily obtained. As $\zeta$ tends to zero, $g$ should vanish. The linearization yields the asymptotic expression as

$$g = \zeta^{3/2} \exp \left( - \frac{1}{4\zeta} \right), \quad 0 < \zeta << 1, \quad (10)$$

so that $f$ is given by

$$f = \frac{\theta^{3/2}}{X^2} \exp \left( - \frac{X^2}{4\theta} \right), \quad 0 < \theta << X^2. \quad (11)$$

This suggests that $f$ steps up very smoothly at the wavefront $\theta = 0$.

For $\zeta>>1$, on the other hand, the similarity solution $g=-\zeta$ to the hyperbolic equation is expected asymptotically. Thus we have

$$g = -\zeta + \frac{4\sqrt{\zeta}}{3\sqrt{\pi}} + \frac{4}{9\pi} \cdot \frac{1}{3} + ..., \quad 1 << \zeta, \quad (12)$$

so that $f$ is given by

$$f = -\frac{\theta}{X} + \frac{4\sqrt{\zeta}}{3\sqrt{\pi}} + \left( \frac{4}{9\pi} \cdot \frac{1}{3} \right) X^+ ..., \quad X^2 << \theta, \quad (13)$$

These asymptotic solutions are used in integrating (4) for $X<<1$ in the sector region $0<\theta<X$. Figure 3 shows how rapidly the initial negative step spreads out by the $X$-steps 0.02 up to $X=0.5$. Comparing with the solution $f=-\theta/X$ to the hyperbolic equation, for example, at $\theta=X=0.5$, the effect of wall friction is evident.

**Gaussian shaped pulses: $F(\theta) = \mp \exp(-\theta^2)$**

For these conditions, $\delta$ cannot be set equal to unity because the pulse has its own characteristic scale in $\theta$. The definition of $\delta$ means that a larger value of $\delta$ corresponds relatively to a smaller value of the nonlinearity $\epsilon$. For such a large value as $\delta=1$, no shock waves are formed and the initial pulses diffuse with the long tails producing behind them. For $\delta=0.1$, however, Figs.4 and 5 show how the smooth initial pulses steepen to evolve into the shock waves up to $X=10$ by the $X$-steps 0.5. Incidentally, the shock formation points in Figs.4 and 5 are at $X=1.255$ and $X=1.279$, respectively.

Without the wall friction, the positive pulse evolves into the triangular shape with the discontinuity at the wavefront [5]. The negative pulse behaves antisymmetrically with the positive one with respect to the origin. Because of the fractional derivative, this antisymmetry breaks down to yield the long tails behind the pulses in both cases. As $X$ increases, the tail becomes pronounced and far-reaching backward. The appearance of the tail is one of the characteristics of the fractional derivative. Although, of course, it is due to the hereditary effect, it also results from the fact that (1) conserves the total initial area under evolution [4]. In this respect, (1)
should be compared with the equation with the right-hand side of (1) replaced by a simple dissipation $-\delta f$.

Figure 4. Evolution of the positive pulse up to $X=10$ by the $X$-steps 0.5 for $\delta=0.1$.

Figure 5. Evolution of the negative pulse up to $X=10$ by the $X$-steps 0.5 for $\delta=0.1$.

CONCLUSION

We have investigated the effect of wall friction on evolution of nonlinear acoustic waves in a long pipe. It is found that the derivative of order $1/2$ fails to balance with the nonlinear steepening unlike the second-order one in Burgers equation. But because its effect accumulates, it affects the evolution in a global fashion. This can be clearly seen in the evolution from the positive step. If Burgers equation is solved under the same condition, the initial step is smoothed out locally into Taylor's shock profile eventually but the global shape of the step profile is unchanged. The emergence of the long tails in the pulses shows that the hereditary effect is left globally behind the main wave. Hence it is concluded that the wall friction exhibits the dissipation not encountered in the propagation through a free space and that it plays a key role in pursuing quantitatively a long-time evolution in the far field.

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REFERENCES


