Mass, momentum, and energy transfer by the propagation of acoustic solitary waves

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This paper considers the mass, momentum, and energy transfer accompanied by the propagation of the acoustic solitary wave in a gas-filled tube. As was demonstrated previously [J. Acoust. Soc. Am. 99, 1971–1976 (1996); Phys. Rev. Lett. 83, 4053–4056 (1999)], the propagation of the solitary waves is made possible by connecting a periodic array of Helmholtz resonators axially with the tube. The solitary wave can convey the mass, momentum, and energy steadily with a constant speed that is subsonic but nearly equal to the linear sound speed. It is emphasized that the quantities transferred are of first order in magnitude. Formulating the basic equations in the conservation form, the total amount of the mass, momentum, and energy transferred is obtained by using the solitary-wave solutions. It has the upper bounds determined by the limiting solitary wave, which are proportional to the size of the resonator and the inverse of its natural frequency. In evaluating the energy transfer, a clear distinction should be made between the gas-dynamic energy and the acoustic energy. The former, which contains the first-order quantity whereas the latter begins with the quadratic ones, is to be used to determine the energy transfer in the form of heat (internal energy) associated with the mass transfer. © 2000 Acoustical Society of America. [S0001-4966(00)02405-X]

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INTRODUCTION

It has recently been revealed, not only in theory but also in experiments, that acoustic solitary waves can propagate in a gas-filled and rigid tube by axially connecting with it a periodic array of Helmholtz resonators.1–3 The solitary wave is a compressive pulse localized both spatially and temporally and is propagated steadily with a constant speed slower than linear sound speed \( a_0 \) (i.e., subsonic), but faster than a threshold value \( a_0/(1 + \kappa/2) \) where \( \kappa \approx 1 \) is a small parameter to measure the size of the resonator. As the speed tends to the upper bound \( a_0 \), the height of the solitary wave increases to approach the limiting value. The excess pressure in the tube corresponding to this height is given by \( 8\gamma\delta/3(\gamma+1) \) relative to the pressure in equilibrium, \( \gamma \) being the ratio of specific heats. The solitary wave in this case is called the “limiting solitary wave.” On the other hand, as the speed tends to threshold, the height decreases and then the solitary wave approaches the Korteweg–de Vries soliton asymptotically.

It is a remarkable property that the solitary waves can be steadily propagated without any change of form. Although no dissipative effects are assumed, their propagation should be compared with that of nonlinear acoustic pulses in a tube without an array of resonators. It is usually the case that they evolve into shocks and decay quickly by nonlinearity even in the lossless limit. But when the array is connected, there arises a weak but pure dispersion, which can now compete with the nonlinearity to yield steady propagation. This persistent propagation implies a steady transfer of physical quantities. Because the solitary wave is a compressive pulse, an increase occurs in density and temperature of the gas, obeying the adiabatic relation. At the same time, the gas moves in phase with the pressure pulse in the direction of propagation, although the speed of the gas itself is much smaller than the wave speed. Thus mass, momentum, and energy transfer can take place. The purpose of this paper is to examine how much of these quantities can be transferred by the propagation of the solitary wave.

In evaluating the energy transfer, one should distinguish the “gas-dynamic energy” as a fluid and the acoustic energy. The gas-dynamic energy consists of the internal (thermal) energy and the kinetic energy. It involves the first-order quantity, which does not vanish but remains when the solitary wave is integrated over the entire space or time. By contrast, the acoustic energy is defined as the sum of the potential energy stored by the excess pressure and the kinetic energy, and both are basically quadratic quantities. The conservation equation of the acoustic energy can be derived legitimately from the equation of the gas-dynamic energy by using the conservation equation of mass to eliminate the first-order terms. But use of the acoustic energy has a meaning only if a mean of the first-order quantity vanishes and then a quadratic relation, although small, is required. Since the integral of the first-order quantity does not vanish in the present case, it is the gas-dynamic energy that should be used in calculating the lowest energy transfer.

In what follows, the basic equations are recapitulated in Sec. I and then rewritten in the conservation form in Sec. II. The solitary-wave solutions are briefly summarized in Sec. III for use to evaluate the mass, momentum, and energy transfer. Section IV is devoted to the explicit calculations of the quantities transferred and to some discussion.
of the cavity, and \( L \) and the integral is taken along the periphery of the cross-section of the tube, \( ds \) being a line element.

**I. BASIC EQUATIONS**

We start with reconsidering the basic equations used already in the previous paper, which will be referred to as Ref. 1. Figure 1 illustrates a tube to which identical Helmholtz resonators are connected in array with equal axial spacing. Since propagation of the solitary wave is a steady phenomenon, we ignore all dissipative effects due to viscosity and heat conduction. No account is taken of the boundary layer which will develop on the tube wall. The phenomenon is one-dimensional (plane) and uniform over the cross-section of the tube. Although the array of resonators gives rise to three-dimensional disturbances, the deviation from the plane wave remains small if the resonator is small in the sense that a cavity’s volume \( V \) is small enough compared with a tube’s volume per axial spacing \( d \) between neighboring resonators. This smallness is measured by a parameter \( \kappa \) defined by \( V/Ad \), \( A \) being a cross-sectional area of the tube. When the inner surface of the tube is uniformly lined with many resonators, \( \kappa \) is taken as \( NV/A \) where \( N \) represents the number density of the resonators per unit axial length. In any case, we make the continuum approximation for the distribution of the resonators. It enables us to average their effects over axial length, on the basis of the assumption that the axial spacing is much smaller than a typical acoustic wavelength.

In formulating the problem, deviations from the plane wave are assumed to be so small that the equations averaged over the cross-section of the tube may effectively be used. We retain only terms of the first-order deviation and ignore all higher-order ones. Then the equation of continuity is given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = \frac{1}{A} \oint \rho v_n \, ds,
\]

where \( \rho(x,t) \) and \( u(x,t) \) represent, respectively, the mean density and axial velocity of the gas over the cross-section of the tube, which depend on the axial coordinate \( x \) and time \( t \). The right-hand side represents the mass flux from the resonators to the tube or vice versa, where \( v_n \) denotes the velocity component normal inward to the inner surface of the tube and the integral is taken along the periphery of the cross-section at \( x \) (see Fig. 1). In the present context, \( v_n \) vanishes on the inner surface except at an orifice where the tube opens to the resonator’s throat. The quantity \( v_n \) gives rise to the small deviation from the plane wave. The right-hand side is small enough in comparison with the terms on the left-hand side and only the lowest term in \( v_n \) will be retained, whereas the nonlinear terms associated with variations in the tube are taken into account.

When the throat of the resonator is connected normal to the axis of the tube, the equation of motion in the axial direction is kept intact as the Euler equation as follows:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x},
\]

where \( p(x,t) \) denotes the mean pressure in the tube. Since all dissipative effects are ignored and both tube and resonators are assumed to be thermally insulated from the outside, no heat is generated or conducted in the gas. Then the equation of energy is given in terms of the mean entropy \( S(x,t) \) per unit mass as follows:

\[
\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0.
\]

The gas is assumed to obey the equation of state for an ideal gas, which is averaged over the cross-section as follows:

\[
\frac{p}{p_0} = \frac{\rho T}{\rho_0 T_0},
\]

with \( p_0 = R \rho_0 T_0 \), where \( T(x,t) \) denotes the mean temperature and \( R \) is a gas constant; the subscript 0 implies the value in equilibrium. Equation (3) states that the entropy of each gas particle is kept constant while in motion so that the gas is subjected to the adiabatic change. In this process, the pressure is expressed in terms of the density as follows:

\[
\frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^{\gamma},
\]

with \( \gamma = c_p/c_v \), where \( c_p \) and \( c_v \) are the specific heats at constant pressure and volume, respectively, and all the quantities in equilibrium are assumed constant everywhere. Equations (1) to (5) are the basic equations for the gas in the tube.

For the gas in the resonator, we consider equations separately in the cavity and in the throat. Assuming no motion of the gas in the cavity, use is only made of the equation for the conservation of mass. Balancing the rate of change of the mass in the cavity with the mass flux into it, the conservation of mass is given by

\[
V \frac{\partial \rho_c}{\partial t} = B q,
\]

where \( \rho_c(x,t) \) and \( q(x,t) \) denote, respectively, the mean density in the cavity and the mean mass flux density averaged over the cross-section of the throat having the area \( B \). The mass flux density \( q \) is assumed to be uniform along the throat because a typical acoustic wavelength is much longer than the length of the throat so that the gas in the throat may be regarded as being incompressible. Quantity \( q \) also represents the momentum density so the momentum balance of the gas in the throat is governed by
where \( L \) is the length of the throat and \( p_c(x,t) \) is the mean pressure in the cavity. These two equations may appear to be valid even in treating the nonlinear response of the resonator. It is assumed that the response is linear because \( v_n \) is small enough and a typical length of the resonator is much smaller than the wavelength. Then the left-hand side of Eq. (6) may be approximated to be \((V/a_0^2)\partial p_c/\partial t\) to the lowest order in the adiabatic process. Using this and eliminating \( q \) from Eqs. (6) and (7), we obtain the linear equation for the resonator’s response:

\[
\frac{1}{\omega_0^2} \frac{\partial^2 p_c'}{\partial t^2} + p_c' = p',
\]

where \( \omega_0 = \sqrt{B/\rho_0 L V} \) is the natural angular frequency of the resonator and \( p' = p - p_0 \) and \( p'_c = p_c - p_0 \) are the excess pressures; the prime is used henceforth to denote an excess quantity from the equilibrium. When the end corrections are made to the throat, \( L \) is lengthened by 0.82\( r \) on each end, \( r \) being the radius of the throat.

II. CONSERVATION FORM

We now rewrite the equations of continuity, motion, and energy presented in the preceding section into the conservation form.

A. Conservation of mass

Denoting the right-hand side of Eq. (1) by \(-\rho \sigma\), it is given by

\[
\frac{1}{A} \int_A \rho v_n \, ds = -\rho \sigma = -\frac{B}{A} \frac{\partial q}{\partial t}. \tag{9}
\]

Here the last term arises because, multiplying by \( d \) both the numerator and denominator on the first term, \( d(\rho v_n) \) gives the area element on the inner surface of the tube where \( v_n \) vanishes except at the orifices. Using Eq. (6) and the parameter \( \kappa \), Eq. (1) can be rewritten into the conservation form:

\[
\frac{\partial}{\partial t} (\rho + \kappa \rho_u) + \frac{\partial}{\partial x} (\rho u) = 0. \tag{10}
\]

This states that the conservation of the mass of the gas holds if the mass in the cavity is taken into account. Note that the temporal variation of the mass in the throat is negligible because the mass flux over the cross section of the throat is assumed to be uniform along its axis.

B. Conservation of momentum

Next we consider the conservation of the momentum in the axial direction of the tube. Multiplying Eq. (2) by \( \rho \) and Eq. (1) by \( u \), respectively, and adding them, it follows that

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) = -\rho \sigma u. \tag{11}
\]

Since the gas in the resonator has no momentum in the \( x \) direction and the gas flown into (or out of) the resonator loses (or gains) the axial momentum, the conservation of the momentum does not hold. In other words, the resonators play the role of the source or sink of the momentum for the gas in the tube, which appears on the right-hand side of Eq. (11). But if unidirectional propagation of weakly nonlinear waves is concerned just as in the case of the solitary waves, Eq. (11) can also be recast into the conservation form.

The right-hand side of Eq. (11) may be rewritten by using Eqs. (6) and (9) as

\[
-\rho \sigma u = -\frac{A}{V} \frac{\partial p_c}{\partial t} = -\kappa \frac{\partial p_c'}{\partial t} u. \tag{12}
\]

Here \( u \) is related to \( p' \) by \( u = p'/p_0 \rho_0 \) to the lowest order (see Ref. 1, p. 64). Using this and Eq. (8),

\[
-\rho \sigma u = -\kappa \frac{\partial m'_c}{\partial t}, \tag{13}
\]

with \( m'_c \) defined by

\[
m'_c = \frac{1}{2 \rho_0 a_0^3} \left[ \frac{1}{\omega_0^2} \left( \frac{\partial p_c'}{\partial t} \right)^2 + p_c'^2 \right]. \tag{14}
\]

As will be shown later, the momentum density \( m'_c \) is related to the energy density of the resonator. Thus Eq. (11) is expressed in the conservation form as

\[
\frac{\partial}{\partial t} (\rho u + \kappa m'_c) + \frac{\partial}{\partial x} (\rho u^2 + p) = 0, \tag{15}
\]

where \( \rho u + \kappa m'_c \) is the momentum density, while \( \rho u^2 + p \) is the momentum flux density. But it should be emphasized again that Eq. (15) holds only if the unidirectional propagation of weakly nonlinear waves is assumed and the source term is evaluated to the lowest order.

C. Conservation of energy

1. Equation for the total energy

In order to put Eq. (3) in the conservation form, we eliminate \( S \) by using the thermodynamic relation for the gas in motion:

\[
T \frac{DS}{Dt} = \frac{D}{Dt} \frac{D}{Dt} \left[ \frac{1}{\rho} \right] = 0, \tag{16}
\]

where \( D/Dt \) signifies the operator defined by \( \partial/\partial t + u \partial/\partial x \), and \( U(x,t) \) denotes the internal energy per unit mass of the gas. Using Eqs. (1) and (9), Eq. (16) is rewritten as

\[
\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} = -\frac{1}{\rho} \frac{\partial u}{\partial x} \left[ \frac{\partial u}{\partial x} + \sigma \right]. \tag{17}
\]

Multiplying this by \( \rho \), and Eq. (1) by \( U \), respectively, addition of both equations yields

\[
\frac{\partial}{\partial t} (\rho U) + \frac{\partial}{\partial x} (\rho U u) = -\rho \frac{\partial u}{\partial x} \left( \rho + u^2 + p \right) \tag{18}
\]

Besides the internal energy, the gas also has kinetic energy \( u^2/2 \) per unit mass. To write down the equation for the total energy \( E \) as the sum of the internal and kinetic energies, i.e., \( E = U + u^2/2 \), we multiply Eqs. (1) and (2) by \( u^2/2 \) and
\[ \frac{\partial}{\partial t} (\rho E) + \frac{\partial}{\partial x} \left[ (\rho E + p) u \right] = -(\rho E + p) \sigma. \quad (19) \]

Here it is to be noted that because the internal energy has an additive constant, we can specify only an excess quantity \( U' (= U - U_0) \) from the equilibrium value \( U_0 \) arbitrarily chosen. But even if \( U' \) is used, the same form of Eq. (19) still holds, by virtue of Eq. (1), with \( E \) replaced by \( E' (= U' + u^2/2) \) as follows:

\[ \frac{\partial}{\partial t} (\rho E') + \frac{\partial}{\partial x} \left[ (\rho E' + p) u \right] = -(\rho E' + p) \sigma. \quad (20) \]

Next, we consider the energy equation for the gas in the resonator. Because there is no heat flux through the wall, the whole energy of the gas in the resonator consists, under the present assumptions, of the internal energy in the cavity and the kinetic energy in the throat. The rate of change of this energy should be balanced by the energy flown into the resonator per unit time through the orifice, plus the power done by the pressure force acting on the cross-section of the orifice. Since the velocity of the gas in the throat is assumed to be uniform over the cross-section, the conservation of the energy in the resonator is given by

\[ \frac{\partial}{\partial t} \left( V \rho_c U_c + \frac{1}{2} B L \rho w^2 \right) = B (\rho E' + p) w, \quad (21) \]

where \( U_c(x,t) \) and \( w(x,t) \) represent, respectively, the internal energy per unit mass of the gas in the cavity and the velocity of the gas in the throat. In this case as well, \( U_c \) and \( E \) may be replaced by their excess quantities \( U'_c \) and \( E' \) by virtue of Eq. (6) with \( q = \rho w \) as

\[ \frac{\partial}{\partial t} \left( V \rho_c U'_c + \frac{1}{2} B L \rho w^2 \right) = B (\rho E' + p) w. \quad (22) \]

Using \( \sigma = B w / A d \) from Eq. (9), we substitute the right-hand side of Eq. (20) for that of Eq. (22). Then we arrive at the equation for the conservation of the total energy of the gas not only in the tube but also in the resonators:

\[ \frac{\partial}{\partial t} \left[ (\rho E' + \kappa \left( \rho_c U'_c + \frac{1}{2} B L \right) \rho w^2 \right] + \frac{\partial}{\partial x} \left[ (\rho E' + p) u \right] = 0. \quad (23) \]

Here the terms within the first square brackets are the energy density, while the terms within the second brackets are the energy flux density.

2. Equation for the acoustic energy

While Eq. (23) is the equation for the “gas-dynamic” energy, it may be transformed into that for the acoustic energy familiar in acoustics. To do so, the explicit expression of \( U' \) is evaluated. In the adiabatic process, the increment of \( U, \Delta U \), is due solely to the work done on the gas, i.e., \( -p d(1/\rho) \). With the excess pressure \( p' = p - p_0 \), the excess internal energy is integrated to yield

\[ U' = -\frac{p_0}{\rho} + \frac{p_0}{\rho_0} - \int_{p_0}^{p'} d\left( \frac{1}{\rho} \right). \quad (24) \]

The first two terms correspond to the work done for volume change under constant pressure in equilibrium and the third term represents the potential energy stored by the excess pressure. The internal energy \( \rho U' \) per unit volume is given by

\[ \rho U' = \frac{p_0}{\rho_0} \rho' + \rho \int_{p_0}^{p'} \frac{d\rho}{\rho^2}. \quad (25) \]

In order to evaluate \( U' \), we notice the fact that \( U' \) for the ideal gas is determined by the temperature only as \( U' = c_v T' \) with \( T' = T - T_0 \). Then Eq. (4) is employed, \( U' \) is immediately obtained by using \( c_v = R \gamma / (\gamma - 1) \) as

\[ U' = \frac{1}{\gamma - 1} \left( \frac{p}{\rho} - \frac{p_0}{\rho_0} \right). \quad (26) \]

Equating the right-hand sides of Eqs. (24) and (26), the integral can easily be obtained as follows:

\[ \rho \int_{p_0}^{p'} \frac{d\rho}{\rho^2} = p_0 + \frac{p_0}{\gamma - 1} \left( \frac{p}{p_0} - \frac{\gamma p_0}{p} \right). \quad (27) \]

When \( p' / p_0 \) is small, Eq. (27) may be expanded by using the adiabatic relation (5) as follows:

\[ \rho \int_{p_0}^{p'} \frac{d\rho}{\rho^2} = \frac{1}{2 p_0 a_0^2} p'^2 - \frac{2}{6 p_0 a_0^4} p'^4 + \ldots. \quad (28) \]

with \( a_0 = \gamma p_0 / p_0 = \gamma R T_0 \). In a similar fashion, \( \rho_c U'_c \) is given as the expression (25) and the corresponding integral is expanded in the same form as the expression (28) with \( \rho \) and \( p' \) replaced simply by \( \rho_c \) and \( p'_c \), respectively.

Let us now go back to Eq. (23). We use the relation (25) and rearrange \( \rho E' \) into the sum

\[ \rho E' = \frac{p_0}{\rho_0} \rho' + e', \quad (29) \]

where \( e' \) is defined by

\[ e' = \rho \int_{p_0}^{p'} \frac{d\rho}{\rho^2} = \frac{1}{2} \rho u^2. \quad (30) \]

Here \( e' \) starts with quadratic terms. Equation (23) can be expanded around equilibrium, on substitution of the relation (29); however, there arise the linear terms \( p_0 p' / \rho_0 \) from \( \rho E' \), and \( p_0 \rho_c / \rho_0 \) from \( \rho_e U'_c \), while there arises \( p_0 p u / \rho_0 \) from \( (\rho E' + p) u \). But these terms drop out automatically by virtue of Eq. (10). As for the contribution from the resonators, we have only to take account of the quadratic terms because they behave linearly. Thus Eq. (23) is reduced to the following form:

\[ \frac{\partial}{\partial t} (e' + \kappa e') + \frac{\partial}{\partial x} e' = 0, \quad (31) \]

with
\[ e'_c = \frac{1}{2\rho_0 a_0^2} \rho c^2 + \frac{1}{2} \frac{BL}{V} \rho_0 w^2, \]  
(32)

and

\[ j' = (e' + p') u. \]  
(33)

This is the acoustic energy equation which holds among quadratic as well as higher-order quantities. If only the quadratic terms are concerned, it is nothing but the well-known equation in the linear acoustics. According to the convention, \( e' \) (and also \( e'_c \)) may be called the acoustic energy density, while \( j' \) may be called the acoustic energy flux (or simply acoustic intensity in the linear case). When \( w \) in \( e'_c \) is replaced with \((V/\rho_0 a_0^2)\partial p'/\partial t\) by Eq. (6), we find that \( e'_c \) is related to \( m'_c \) through \( e'_c = a_0 m'_c \). It should be remarked that the acoustic energy density and the acoustic energy flux differ from the gas-dynamic ones defined in Eq. (23). The gas-dynamic quantities \( \rho E' \) and \( \rho U'_c \) contain the first-order terms \( \rho p'/\rho_0 \) and \( \rho_0 p'/\rho_0 \). As usually is the case with harmonic waves in linear acoustics, the time average of a quantity vanishes. Then, only the quadratic quantities have physical significance. This is the reason the acoustic energy is used. But when the time average of a quantity does not vanish, as in the case with solitary waves, we should employ the gas-dynamic quantities given in Eq. (23) to discuss thermal energy. Yet the acoustic energy density and flux give the quadratic quantity to be conserved.

### III. SOLITARY-WAVE SOLUTIONS

In this section, we present the explicit form of the solitary-wave solutions obtained in Ref. 2. Solitary waves exist as steady progressive-wave solutions to the following system of equations for \( f(\theta, X) \) and \( g(\theta, X) \):

\[ \frac{\partial f}{\partial X} + f \frac{\partial f}{\partial \theta} = -K \frac{\partial g}{\partial \theta}, \]  
(34)

\[ \frac{\partial^2 g}{\partial \theta^2} + \Omega g = \Omega f. \]  
(35)

With the small parameter \( \epsilon \) (0<\(\epsilon\)\(\leq\)1) specifying the weakness of the nonlinearity, \( \epsilon f \) and \( \epsilon g \) \([f \sim g \sim O(1)]\) represent the excess pressures \([\gamma(y+1)/2\gamma p'/p_0 \) and \([\gamma(y+1)/2\gamma p'/\rho_0 \) in the tube and in the cavity, respectively. The independent variables \( X \) and \( \theta \) are, respectively, the far-field coordinate \( \epsilon e_x t/a_0 \) associated with the nonlinearity, and the retarded time \( \epsilon(t-x/a_0) \) in a frame moving with the sound speed \( a_0 \), where \( \omega \) is a typical angular frequency. The parameters \( K \) and \( \Omega \) designate the ratio of the smallness of the resonator, and the ratio of the resonator’s natural angular frequency to the typical frequency. They are defined as follows:

\[ K = \frac{\kappa}{2 \epsilon} \]  
and

\[ \Omega = \left( \frac{\omega_0}{\omega} \right)^2. \]  
(36)

In Ref. 2, the solutions have been obtained after effecting the replacement \( K = \Omega = 1 \):

\[ (f, g) = (K\tilde{f}, K\tilde{g}) \]  
and  
\[ (X, \theta) = (X/K\sqrt{\Omega}, \theta/K\sqrt{\Omega}). \]  
(37)

The solitary-wave solutions are obtained by looking for the solutions to Eqs. (34) and (35) in the form of \( \tilde{f} = \hat{f}(\xi) \) and \( \tilde{g} = \hat{g}(\xi) \) where \( \xi = \theta - sX, s \) being a parameter. The variable \( \xi \) is related to \( x \) and \( t \) through

\[ \xi = \omega_0 \left[ t - \frac{x}{a_0} \left( 1 + \frac{\kappa s}{2} \right) \right]. \]  
(38)

From this, the propagation speed of the solitary wave is given by \( a_0(1 + \kappa s/2)^{-1} = a_0(1 - \kappa s/2 + O(s^2)) \). It was demonstrated that the solitary-wave solution can exist only if \( 0 < s < 1 \). This can be expressed as follows:

\[ 4 \tan^{-1} \sqrt{f_+ - f_-/f_+ - f_-} = 2s \begin{array}{c} \left[ -\sqrt{f_+ - f_-} - \sqrt{f_+ + (f_+ - f_-)^2} \right. \\ \left. \sqrt{(f_+ - f_-)^2} \right] \end{array} \]  
(39)

where \( f_\pm \) are given by

\[ f_\pm = -2 \left( s - \frac{2}{3} \right) + \sqrt{\frac{4}{3} s + \frac{16}{9}}, \]  
(40)

with the sign \( \pm \) vertically ordered. For later use, we note that the solution (39) is obtained by integrating the following differential equation

\[ \left( \frac{df}{d\xi} \right)^2 = \frac{\hat{f}^2(f_+ - \hat{f})(\hat{f} - f_-)}{4(f + s)^2}, \]  
(41)

together with such boundary conditions as the undisturbed state far ahead of propagation, \( \tilde{f} \to 0 \) as \( \xi \to -\infty \) (i.e., \( x \to \infty \)), and the peak at the origin, \( df/d\xi \to 0 \) at \( \xi = 0 \). When \( f \) is solved, then \( g \) is available through the algebraic relation

\[ \hat{g} = \hat{f}^2 + sf. \]  
(42)

Although the solution (39) appears to have a very complicated form, \( \tilde{f} \) graphically takes a simple waveform of a pulse symmetric with respect to the peak at \( \xi = 0 \) and decayed out exponentially as \( |\xi| \) increases. The explicit waveforms of \( \tilde{f} \) and \( \hat{g} \) are displayed for some values of \( s \) (see Fig. 3 of Ref. 2). The heights in \( \tilde{f} \) and \( \hat{g} \) given, respectively, by \( f_+ \) and \( g_+ \left( = f_+^2/2 + sf_+ \right) \) increase monotonously as \( s \) decreases in the range \( 0 < s < 1 \).

Here we refer to the limiting behavior of the solitary waves as \( s \to 0 \) and \( s \to 1 \). As \( s \) tends to vanish, \( \tilde{f} \) and \( \hat{g} \) approach, respectively,

\[ \tilde{f} = \frac{8}{3} \cos^3 \frac{\xi}{4} \]  
and

\[ \hat{g} = \frac{32}{9} \cos^4 \frac{\xi}{4}, \]  
(43)

for \( -2\pi \leq \xi \leq 2\pi \) and \( \tilde{f} = \hat{g} = 0 \) for \( |\xi| > 2\pi \). Because these solutions lose the regularity at \( |\xi| = 2\pi \), they cannot be achieved and must be regarded as the limiting solutions as \( s \to 0 \). Thus these solutions are called the limiting solitary
wave. On the other hand, as \( s \to 1 \), \( \tilde{f} \) and \( \tilde{g} \) approach the soliton solution of the Korteweg–de Vries equation asymptotically. They are given by

\[
\tilde{f} = \tilde{g} = \alpha \text{sech}^2 \sqrt{\frac{\alpha}{12}} \zeta,
\]

(44)

with \( s = 1 - \alpha/3 \) \((0 < \alpha < 1)\) and \( \zeta = \bar{\theta} - \bar{X} + \alpha \bar{X}/3 \). It follows from the relation (42) that \( \tilde{g} \) tends to \( \tilde{f} \) as \( s \to 1 \) (i.e., \( \alpha \to 0 \)).

Finally, we make the following points. Parameter \( \epsilon \) has been introduced to designate the order of the excess pressures in the tube and in the cavity, referenced to \( p_0 \), and used as the formal expansion parameter to derive the system of Eqs. (34) and (35). It is assumed small but left unspecified so far. By the definition (36) and the replacement (37), \( \epsilon \tilde{f} \) and \( \epsilon \tilde{g} \) turn out to be \( \kappa \tilde{f}/2 \) and \( \kappa \tilde{g}/2 \), respectively, where \( \tilde{f} \) and \( \tilde{g} \) are regarded as being of order unity. This suggests that the order of \( \epsilon \) for the solitary waves is stipulated by the smallness of \( \kappa \). In fact, the peak excess pressures \( \Delta \tilde{p}' \) and \( \Delta \tilde{p}_c' \) in the tube and in the cavity are given, respectively, by \( \Delta \tilde{p}'/p_0 = \kappa \gamma_f \tilde{f}/(\gamma + 1) \) and \( \Delta \tilde{p}_c'/p_0 = \kappa \gamma_g \tilde{f}/(\gamma + 1) \). Second, it is only within the lowest order of \( \kappa \) that the present theory can specify the excess pressures. In order to calculate them correctly up to the second order of \( \kappa \), Eqs. (34) and (35) should be modified to include the higher-order terms of \( \kappa \) (or \( \epsilon \)). This process would be straightforward but laborious.

Within the present lowest approximation, the other quantities are given in terms of \( \tilde{f} \) and \( \tilde{g} \) as follows:

\[
\frac{p'}{p_0} = \gamma \frac{\tilde{f}}{\tilde{g}} = \frac{\gamma}{\gamma - 1} \left( \frac{T'}{T_0} \right) = \frac{\gamma w}{a_0} = \frac{\kappa \gamma}{\gamma + 1} \tilde{f}.
\]

(45)

These lowest relations, except for the last but one, are simply the linear relations derived from the equation of state and the adiabatic relation. On the other hand, the quantities associated with the resonators are also calculated as follows:

\[
\frac{p_c'}{p_0} = \gamma \frac{\tilde{f}}{\tilde{g}} = \frac{\kappa \gamma}{\gamma + 1} \tilde{g} \quad \text{and} \quad \frac{w}{a_0} = \frac{\kappa}{\gamma + 1} \left( \frac{a_0}{L \omega_0} \right) \frac{d \tilde{g}}{d \zeta}.
\]

(46)

Here the second relation is derived from the linearized version of Eq. (6): \( \rho c\partial \tilde{p}_c' / \partial t = B \rho_0 w \) with \( \rho_c' = p'_c/a_0^2 \), and the definition (38). The factor \( a_0/L \omega_0 \) implies the ratio of a wavelength \( a_0/\omega_0 \) (divided by \( 2\pi \)) corresponding to the natural frequency of the resonator (not a wavelength of the acoustic waves concerned) to the throat’s length \( L \). It is assumed that this ratio is large but the small parameter \( \kappa \) makes \( |w/a_0| \) much smaller than unity.

**IV. CALCULATIONS OF MASS, MOMENTUM, AND ENERGY TRANSFER**

With the explicit solutions in the preceding section, we now calculate the total amount of the mass, momentum, and energy transfer accompanied with the propagation of the solitary wave. Integrating Eqs. (10), (15), and (23) over the whole region of \( x (-\infty < x < \infty) \) and using the fact that all excess quantities vanish exponentially as \( |x| \to \infty \), we have

\[
\frac{d}{dt} \int_{-\infty}^{\infty} (\rho' + \kappa \rho_c') dx = 0,
\]

(47)

and

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \left[ \rho E' + \kappa \left( \rho_c U_c' + \frac{1}{2} \frac{B L}{V} \rho w^2 \right) \right] dx = 0.
\]

(49)

Thus it is found that the integrals of the respective densities are conserved with respect to \( t \).

In Eqs. (47) to (49), the terms associated with the resonators are smaller by the order of \( \kappa \). To evaluate them correctly, \( \rho', \rho u, \) and \( \rho E' \) should be specified up to the order of \( \kappa^2 \) inclusive. With the present approximation, however, we cannot evaluate the integrals up to this order and must be satisfied with the lowest-order transfer. This means the neglect of the terms with \( \kappa \) in the integrals.

By the lowest relations (45), the total excess mass is calculated by

\[
\int_{-\infty}^{\infty} \rho' dx = \frac{\rho_0}{\gamma} \int_{-\infty}^{\infty} \frac{p'}{p_0} dx = \frac{\rho_0}{\gamma + 1} \int_{-\infty}^{\infty} \kappa \tilde{f} dx,
\]

(50)

where \( t \) is held fixed. Changing the variable in the integral from \( x \) to \( \zeta \), it follows that

\[
\int_{-\infty}^{\infty} \kappa \tilde{f} dx = \frac{\kappa a_0}{\omega_0 (1 + \kappa \sigma / 2)} \int_{-\infty}^{\infty} \tilde{f} d \zeta + O(\kappa^2).
\]

(51)

Introducing the definition,

\[
\int_{-\infty}^{\infty} \tilde{f} d \zeta = I_1(s),
\]

(52)

the total excess mass is written as

\[
\int_{-\infty}^{\infty} \rho' dx = \frac{\kappa}{\gamma + 1} \left( \frac{\rho_0 a_0}{\omega_0} \right) I_1(s).
\]

(53)

In a similar way, the total amount of the lowest momentum transfer is calculated by the following integral:

\[
\int_{-\infty}^{\infty} \rho_0 u dx = \frac{\kappa}{\gamma + 1} \left( \frac{\rho_0 a_0}{\omega_0} \right) I_1(s).
\]

(54)

It is found that since solitary waves are propagated with the speed close to the sound speed, the amount of the momentum transfer is given, to the lowest order, by that of the mass transfer times \( a_0 \).

We next calculate the energy transfer in the form of the internal energy. The lowest term in the gas-dynamic energy density \( \rho E' \) is given by \( (\rho_0/p_0) \rho' \) in the expression (25). Because \( (\rho_0/p_0) \rho' = \rho_0 T' / [(\gamma - 1) T_0] = c_e \rho_0 T' \) by the relations (45) and \( R = c_e (\gamma - 1) \), the total amount of the lowest internal energy conveyed is given by

\[
\int_{-\infty}^{\infty} \rho_0 c_e T' dx = \frac{\kappa}{\gamma + 1} \left( \frac{\rho_0 a_0}{\omega_0} \right) I_1(s).
\]

(55)

Note that the quantities (53), (54), and (55) are given per unit cross-sectional area of the tube. Indeed they have been cal-
culated by integrating the respective densities at a time \( t \) over the entire region of \( x \). Since the solitary waves are localized both spatially and temporally, we note that they can also be obtained by integrating the respective fluxes at a position \( x \) over \( t \) from \( t = -\infty \) to \( t = \infty \). For example, Eq. (53) is available by integrating the lowest mass flux density \( \rho_0 \mu \) in Eq. (10) over \( t \).

We calculate the acoustic energy of the solitary wave by integrating the energy density \( e' \) over \( x \). Because \( p' = \rho_0 a_0 \mu \) to the lowest order, the potential energy is equal to the kinetic energy and the total amount is given by

\[
\int_{-\infty}^{\infty} e' \, dx = \int_{-\infty}^{\infty} \frac{p' r^2}{\rho_0 a_0} \, dx.
\]

To calculate the integral, we need the following integral of \( \bar{f}^2 \) defined by

\[
\int_{-\infty}^{\infty} \bar{f}^2 \, d\bar{z} = I_2(s).
\]

Then the acoustic energy is given, to the lowest order, by

\[
\int_{-\infty}^{\infty} e' \, dx = \frac{2}{\gamma} \left( \frac{\rho_0 a_0}{\omega_0} \right) I_2(s).
\]

This is equal to the total amount of the acoustic energy flux which passes a location \( x \).

Thus the problem is reduced to evaluation of the two integrals \( I_1(s) \) and \( I_2(s) \). With the solution (39), it appears to be very complicated to execute the integration. But we can achieve this easily if the differential equation (41) is used. Because \( \bar{f} \) is symmetric with respect to \( \bar{z} = 0 \), we have

\[
I_1(s) = 2 \int_{0}^{\infty} \bar{f} \, d\bar{z}.
\]

We then change the integral with respect to \( \bar{z} \) to that with respect to \( \bar{f} \) by noting that \( \frac{df}{d\bar{z}} < 0 \) for \( \bar{z} > 0 \). In fact, it follows that

\[
I_1(s) = 2 \int_{0}^{\infty} \frac{d\bar{z}}{df} \, df = 4 \int_{0}^{\infty} \frac{(\bar{f} + s)}{\sqrt{\bar{f} - f} (\bar{f} - f)} \, df.
\]

This integral can be executed analytically as

\[
I_1(s) = 8 \sqrt{s(1-s)} + \frac{16}{3} (1 - 4s) \cos^{-1} \left( \frac{-2 + 3s}{\sqrt{4 - 3s}} \right).
\]

where \( \cos^{-1} z \) \(( -1 \leq z \leq 1)\) is defined to take the principal value between 0 and \( \pi \). In a similar fashion, \( I_2 \) can also be integrated analytically as

\[
I_2(s) = 16 \sqrt{s(1-s)} + \frac{8}{3} (1 - s)(4 - 3s) \cos^{-1} \left( \frac{-2 + 3s}{\sqrt{4 - 3s}} \right).
\]

Figure 2 shows the graph of \( I_1(s) \) and \( I_2(s) \) versus \( s \) where \( 0 \leq s < 1 \). As \( s \) increases, both integrals decrease monotonically to vanish at \( s = 1 \). As \( s \) tends to vanish, \( I_1(s) \) and \( I_2(s) \) approach \( 16\pi/3 \) and \( 32\pi/3 \), respectively. The limiting values can also be calculated by using the solutions (43). As \( s \) tends to unity, on the other hand, \( I_1(s) \) and \( I_2(s) \) vanish as \( 12(1-s)^{1/2} \) and \( 24(1-s)^{3/2} \), respectively.

With \( I_1(s) \) and \( I_2(s) \) available, we now obtain explicitly the total amount of the mass, momentum, and energy transferred, respectively. Since \( I_1(s) \) is bounded, the quantities (53) to (55) have the upper bounds, which are given, respectively, by

\[
0 < \int_{-\infty}^{\infty} \rho' \, dx < \frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0}{\omega_0} \right),
\]

\[
0 < \int_{-\infty}^{\infty} \rho_0 \mu \, dx < \frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0}{\omega_0} \right),
\]

\[
0 < \int_{-\infty}^{\infty} \rho_0 c_0 T' \, dx < \frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0}{\omega_0} \right).
\]

Similarly, the acoustic energy is also bounded as follows:

\[
0 < \int_{-\infty}^{\infty} e' \, dx < \frac{32 \pi \kappa^2 \gamma}{3(\gamma + 1)^2} \left( \frac{\rho_0 a_0}{\omega_0} \right).
\]

One solitary wave cannot convey mass, momentum, and energy greater than these limiting values. It is worth noting that the amount conveyed is increased as \( \omega_0 \) becomes low.

Using these formulas, we calculate the upper bounds of the quantities transferred in the tube used for the experiments. For this tube, the parameter \( \kappa \) takes the value 0.197 and the natural frequency is 238 Hz with the end corrections. Then the upper bounds in the quantities (63) to (65) are given by the following values:

\[
\frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0}{\omega_0} \right) = 3.8 \times 10^{-1} \text{ kg/m}^2,
\]

\[
\frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0^2}{\omega_0} \right) = 1.3 \times 10^2 \text{ N\cdots/m}^2,
\]

and

\[
\frac{16 \pi \kappa}{3(\gamma + 1)} \left( \frac{\rho_0 a_0}{\omega_0} \right) = 3.1 \times 10^1 \text{ kJ/m}^2.
\]
where \( \rho_0 = 1.2 \text{ kg/m}^3 \), \( p_0 = 1.0 \times 10^5 \text{ Pa} \), \( \gamma = 1.4 \), and \( a_0 = 340 \text{ m/s} \). On the other hand, the acoustic energy is smaller than that in the gas-dynamic one by \( 2\kappa\gamma(\gamma+1) \). In the present case, the upper bound of the total amount of the acoustic energy is about 23% of that of the thermal energy and is given by 7.2 kJ/m².

V. CONCLUSION

The mass, momentum, and energy transfer due to acoustic solitary waves propagating through a lossless and ideal gas in a tube with a periodic array of Helmholtz resonators have been examined. The transfer occurs steadily with a constant speed that is subsonic but nearly equal to the linear sound speed. The total amount of mass, momentum, and energy transfer has been calculated to the lowest order of the small parameter \( \kappa \) measuring the size of the resonator. It has upper bounds determined by the limiting solitary wave. It should be emphasized that the transfer takes place in the first-order quantity so the amount is proportional to \( \kappa \). It may also be worth noting that the total amount increases in proportion to the inverse of the natural angular frequency of the resonator \( \omega_0 \). Finally it is expected that the phenomenon of the steady transfer due to the solitary waves will be exploited in engineering devices such as heat pipes, heat engines, or heat pumps.

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