Experiments of high-amplitude and shock-free oscillations of air column in a tube with array of Helmholtz resonators

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This experimental study is made to verify the weakly nonlinear theory for high-amplitude and shock-free oscillations of an air column developed in the previous paper [Sugimoto et al., J. Acoust. Soc. Am., 114, 1772–1784 (2003)]. The experiments use a new tube and resonators designed so as to not only avoid higher harmonic resonances and evanescences but also reduce the values of the coefficient of $Q$ in the amplitude equation, and a rubber diaphragm sandwiched by circular plates to drive the air column. The steady-state pressure field in the tube and in the cavities of the resonators is measured, from which Fourier coefficients are obtained. In spite of nonlinearity, higher harmonics are suppressed significantly as designed, and the frequency response measured shows quantitatively good agreement with the one predicted up to about 170 dB (SPL). The first harmonics and the nonoscillatory component in the pressure field are well predicted, though the second harmonics show a quantitative discrepancy with the theory. In view of the good agreement of the frequency response, it is concluded that the theory is valid and useful enough to provide guidelines in designing the tube with the array of resonators. © 2005 Acoustical Society of America. [DOI: 10.1121/1.1929237]

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I. INTRODUCTION

It is known that high-amplitude and shock-free oscillations of an air column are achieved in “dissonant tubes.” In fact, oscillations with amplitude higher than 10% of equilibrium pressure (about 170 dB in the sound pressure level) are generated by exciting the air column resonantly. The term “dissonant” used in contrast to “consonant” means such a tube that resonance frequencies of the air column are not ordered as integral multiples of the fundamental one. At present, there are two methods available to make a tube dissonant, one being to make a cross section of the tube non-uniform axially and the other to exploit wave dispersion.

If dispersion is present, the phase velocity differs from a sound speed and depends on a frequency. Thus a tube becomes naturally dissonant. The dispersion can be introduced by connecting an array of side branches in any form along a tube. Sugimoto et al. have made a tube with a periodic array of Helmholtz resonators (called simply resonators hereafter) to demonstrate the annihilation of shocks and the generation of high-amplitude oscillations. Later they have developed a weakly nonlinear theory to seek the pressure and flow field in the tube and the resonators to derive the amplitude equation, from which the frequency response is obtained. But there is a significant discrepancy in the frequency response between the results measured and predicted. One reason for this discrepancy turns out to be that the tube and resonators do not meet the conditions required by the theory. The purpose of this paper is thus to verify the theory by using a new tube and resonators designed in light of the results of the theory.

When the array of resonators is connected to a tube of uniform cross section, there arises an infinite number of pairs of resonance frequencies. In addition, a new mode appears at a frequency just above the resonance frequency of the Helmholtz resonator in which the air column oscillates in unison without exhibiting any axial structure. Focusing on the resonance at the lowest frequency with one pressure node in the middle of tube, the theory assumes no harmonic resonances and evanescences which will occur, respectively, where frequencies of higher harmonics coincide with one of the resonance frequencies of the tube or with the resonance frequency of the Helmholtz resonator.

Although no second harmonic resonance occurs in a strict sense, a situation close to it would be plausible. Then the coefficient $Q$ (not the quality factor) of the nonlinear term in the amplitude equation takes a large value so that the response curve is bent and the peak pressure is suppressed. This is the situation encountered in the previous experiments. Even worse, the frequency of the fifth harmonics happens to coincide with the resonance frequency of the resonator. Such higher harmonic evanescences do not contribute to making the value of $Q$ large. But when they occur, the energy of excitation would be absorbed in the resonators and the peak pressure would also be suppressed. The theory excludes these cases up to the third harmonics. But it is desirable in designing the tube with the array of resonators to avoid them up to any harmonics as much as possible.

In order to generate higher peak pressure, the jet loss of the throat of the resonator should also be reduced. This requires choice of a smaller cavity and a wider throat but has nothing to do with throat length. It turns out that the throat should be lengthened in order to reduce the values of $Q$ as well as of $D$ for the jet loss. The tube and the resonators used...
in the previous experiments are not the ones designed by taking account of the above-mentioned conditions but the existing ones for the experiments of the acoustic solitary waves. In consequence, the frequency response agrees with the theory only up to the peak pressure of a few percent of the equilibrium pressure. Now that these restrictions have been unveiled, a tube and resonators are chosen to avoid them.

As for the method to excite the air column, the previous experiments exploit the bellows mounted at one end of the tube. Although they have a merit of securing the hermetic sealing, nonuniformity in inner diameter gives rise to unignorable effects if a tube is not long enough in comparison with the depth of the bellows. The present experiments use a rubber diaphragm sandwiched by a couple of circular plates, whose center is driven by a linear motor. When the center of the plates is displaced axially, the rubber stretches to form a circular-cone frustum with the plates. This device is different from a plane piston but a flow field near it would be much simpler than that driven by the bellows and closer to the piston. The depth (or height) of the frustum can be related equivalently to the displacement of the piston by equating the volume displaced.

In what follows, Sec. II summarizes the results of the weakly nonlinear theory necessary to comparison with the ones measured. Section III describes the experimental setup. The experiments measure not only the frequency response but also the pressure field in the tube and the resonators. Results of measurements are compared against the theory in Sec. IV and discussions are given in Sec. V. Finally the conclusions are given.

II. SUMMARY OF THE THEORY

A. Outline of formulation

An air column in a tube of radius $R$ and of length $l$ is driven sinusoidally by a plane piston installed at one end of the tube with the other end closed by a flat plate. Identical Helmhotz resonators are connected to the tube in array through the side wall with equal axial spacing $d$. Each resonator consists of the cavity of volume $V$ and the throat of radius $r$ and of length $L$. The volume of each resonator is assumed to be small in the sense that the size parameter $\kappa$, defined as $V/A/d$, is much smaller than unity, where $A$ is the cross-sectional area of the tube. The axial spacing is taken much smaller than a wavelength of oscillations so that the continuum approximation may be applied to smear the discrete distribution of the resonators.

A natural angular frequency of the resonator is given by $\omega_0 = \sqrt{n\mu_0/\rho_0}$ where $n_0$ is the sound speed and $L$ is lengthened to $L_0 = (L + 2 \times 0.82r)$ by taking account of the end corrections. Because the volume of the throat is usually negligibly small in comparison with the one of the cavity, no distinction is made between the volume of the cavity and the total volume of the resonator $V_r = (V + \pi r^2 L)$. But when the volume of the throat is small but not negligible, it is found that $\omega_0$ based on $V_r$ instead of $V$ fits better with a measured one.\(^7\)

The Reynolds number $a_0t_0/\nu$ is so high that effects of viscosity and heat conduction are confined only in a boundary layer on the tube, where $\nu_0$, $\nu$, and $\omega$ denote, respectively, a typical axial speed of air, its kinematic viscosity, and a typical angular frequency of oscillations. Since the boundary layer is thin and the size parameter is small, the main flow in the outside of the boundary layer and except for the neighborhoods of throat orifices may be regarded as being almost one-dimensional. Assuming the ideal gas for the air and the adiabatic relation in the main flow, one-dimensional, nonlinear wave equation (21) in Ref. 6 is derived from the equations of continuity and of axial motions of air averaged over the cross section of the main flow in terms of the velocity potential $\phi(x,t)$, $x$ and $t$ being, respectively, the axial coordinate along the tube and the time, and the overbar used to designate the dimensionless quantities being suppressed. Effects of the boundary layer and the array of resonators are taken into account in the form of a source term in the equation of continuity.

For response of the resonators, the compressibility of air in the throat is ignored because the throat length is much smaller than the wavelength, and the equation of axial motion along the throat is averaged over it by taking account of a boundary layer on the throat wall. For the air in the cavity, no motions are assumed and only the mass balance is required with the adiabatic relation. To simulate the response of the resonator well, it is of vital importance to consider a jet loss at the throat which occurs when the air flows through it. Using the jet loss in a semi-empirical form, the oscillation equation (95) in Ref. 6 for the excess pressure in the cavity $p'_c(x,t)$ is derived with the excess pressure in the tube $p'(x,t)$ as the forcing term. Of course, $p'$ is derived from $\phi$ by the Bernoulli’s theorem.

B. Assumptions of the theory

It is revealed in the lossless linear theory that the natural oscillations of the air column confined in $0 < x < 1$ by flat plates at both ends are given in the dimensionless form as follows:

\[
\begin{align*}
\left[ \begin{array}{c}
\phi \\
p \\
p_c' \\
\end{array} \right] &= \left[ \begin{array}{c}
il \pi \sigma \\
1 \\
s \\
\end{array} \right] \cos[k(x-1)]a e^{i \pi \sigma} + c.c., \\
\end{align*}
\]

(1)

where $\phi$ is normalized by $lu_0$, $p'$ and $p'_c$ by $\rho_0 t_0 a_0$, $x$ and $t$ by $l$ and $l/a_0$, respectively, $\rho_0$ being the density of air in equilibrium; the wavenumber $k$, given by

\[
k^2 = (\pi \sigma)^2(1 + \kappa s)
\]

with

\[
s = \frac{a_0^2}{a_0^2 - \sigma^2},
\]

is chosen to be $m \pi$ ($m = 0, 1, 2, \ldots$) by the boundary conditions at both ends, from which the dimensionless angular frequency $\sigma = \omega l/a_0$ is determined, $\omega$ being a dimensional one; $a$ denotes the complex amplitude, and c.c. implies the complex conjugate to all preceding terms. For a given value

\[
\]

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of $m$, there occurs a pair of the natural angular frequencies $\sigma_m^\pm$ given by

$$\sigma_m^\pm = \frac{1}{\sqrt{2}} \sqrt{(m^2 + (1 + \kappa)\sigma_0^2) \pm \sqrt{(m^2 + (1 + \kappa)\sigma_0^2)^2 - 4m^2\sigma_0^4}}^{1/2},$$

(3)

where $\sigma_0 = (l\omega_0 / \pi a_0)$ denotes the dimensionless natural angular frequency of the resonator. Note that $\pi a_0/l$ is the lowest natural angular frequency of the air column in the tube without the array.

In Fig. 1, the solid curves show the graph of the dispersion relation (2) which stipulates for an arbitrary value of $k$ in the case with $\sigma_0=2.5$ and $\kappa=0.2$, and the closed circles indicate the natural angular frequencies $\sigma_m^\pm$ for $k=m\pi$ ($m =1,2,3,\ldots,6$) selected by imposing the boundary conditions at both ends. By connecting the array of resonators, the dispersion curves are split into two branches, which are separated by the stopping band in the frequency range $\sigma_0 < \sigma < (1+\kappa)^{1/2}\sigma_0$. The lower bound corresponds to the evanescent with $k \to \infty$, while the upper one to the resonance with $k \to 0$ mentioned briefly in Sec. I. For the lowest natural angular frequency $\sigma_1$, the straight line $\sigma = \sigma_1 k/\pi$ is drawn to indicate that all the other frequencies are located off the line so that the tube is dissonant.

Let one end of the air column be driven by a plane piston at $x=x_p = c \cos(\pi \sigma t)$ with $c = X_p/l \ll 1$ and at a frequency close to one of $\sigma_1^* (= \sigma_1)$ where $X_p$ denotes the dimensional displacement amplitude of the piston surface so the air column occupies the region $x_p < x < l$. Then three quantities come into play. One is a parameter specifying the order of detuning of the driving frequency from the natural one, $\Delta \sigma / |\Delta \sigma| \ll 1$, and another is a parameter for the order of oscillations. While this is given by the maximum excess pressure relative to the equilibrium one, it is the acoustic Mach number $\epsilon = (\epsilon a_0 / a_0 \ll 1)$ that is used in the theory. Of course, both parameters are comparable in order. The theory assumes such a situation that

$$\epsilon = \frac{c}{|\Delta \sigma|}.$$  

(4)

In view of the experiments in which $c$, $\Delta \sigma$, and $\epsilon$ are of order $10^{-3}$, $10^{-2}$ and $10^{-1}$, Eq. (4) is satisfied when $c$ is assumed to be of $\epsilon^3$ while $|\Delta \sigma|$ of $\epsilon^2$.

The other quantities are parameters specifying the order of dissipation due to the boundary layers on the tube wall and throat wall. They are given, respectively, by $\delta$ and $\delta_*$ defined as follows:

$$\delta = 2 \left( 1 + \frac{\gamma - 1}{\Pr} \right) \frac{\sqrt{\nu l/a_0}}{R^*} \ll 1, \quad \delta_* = 2 \frac{\sqrt{\nu l/a_0}}{r^*} \ll 1,$$

(5)

where $\gamma$ and $\Pr$ denote the ratio of the specific heats and the Prandtl number, respectively, while $R^*$ and $r^*$ denote the reduced radii defined, respectively, as $R^* = [R/(1 - \pi r^2 R/2A)]$ and $r^* = (r/l_0 c_f)$, $c_f = (l+2r)/L_c$. The values of these parameters are regarded as the quantities of $\epsilon^2$.

The theory assumes no higher harmonic resonances and evanesces up to the third harmonics. If a frequency of the $n$th higher harmonics, $n\sigma_1$, $n$ being an integer greater than unity, coincides with $\sigma_m^\pm (m=0,1,2,\ldots)$, i.e.,

$$(m\pi)^2 = (n\pi \sigma_1)^2 \left[ 1 + \frac{\kappa \sigma_0^2}{\sigma_0^2 - (n\sigma_1)^2} \right],$$

(6)

then $\sigma_m^\pm$ is determined in terms of $\sigma_0$ as

$$\sigma_m^\pm = \frac{1}{2(n^2-1)} \left[ (m^2-1) \pm (m^2 - 1)^2 - 4(m^2 - n^2) \right]$$

$$\times (1-n^2)(\sigma_0^2)^{1/2},$$

(7)

This relation may be transformed into the expression for $\sigma_0^2$ in terms of $\kappa$ as

$$\sigma_0^2 = \frac{1}{2(n^2 - 1)^2} \left[ j_{mn} \pm \sqrt{j_{mn}^2 - 4(1+\kappa)^2 n^2} \right],$$

(8)

with

$$j_{mn} = \frac{(m^2-n^2)^2 - (1+\kappa)(m^2(\nu^2 - 1)^2 + (m^2-n^2)^2)n^2}{(m^2-n^2)(n^2-\nu^2)},$$

(9)

or alternatively into the expression for $\kappa$ in terms of $\sigma_0^2$ as

$$\kappa = \frac{2(m^2-n^2)^2 - (m^2-n^2)^2 n^2 \pm k_{mn}}{2(m^2-n^2)(n^2-\nu^2)\sigma_0^2} - 1,$$

(10)

with

$$k_{mn} = \sqrt{(m^2-n^2)^2 [(m^2-\nu^2)n^2 - 4(m^2-n^2)(n^2-\nu^2)\sigma_0^2]}.$$  

(11)

As far as $\sigma_0^2$ or $\kappa$ thus obtained is positive, they have physical meanings. If $n\sigma_1$ hits $\sigma_0$ for evanescent, $\kappa$ and $\sigma_0^2$ are determined as

$$\sigma_0^2 = \frac{n^2-n^4}{1 - (1+\kappa)n^2} \quad \text{or} \quad \kappa = \frac{(n^2-1)(n^2-n^4)}{n^2\sigma_0^2}.$$  

(12)

In order to achieve higher pressure oscillations in reality, it is desirable to avoid these resonances and evanesences beyond
the third harmonics in designing the tube with the array of resonators. Even if these conditions were not satisfied exactly, it is desirable to depart from them as much as possible.

C. Results of the theory

The solutions for the velocity potential in the tube and the pressure in the cavities of the resonators are sought in the asymptotic expansion with respect to $\varepsilon$ by the method of multiple (double) time-scales up to the order of $\varepsilon^2$. The main outcome is the amplitude equation which determines a temporal behavior of the complex amplitude $\alpha$ for the dimensionless excess pressure in the tube. The equation is derived from the boundary conditions of $\varepsilon^2$ as [see Eq. (100) in Ref. 6 with the correction ($\sigma_1$ on the right-hand side to be removed)]

$$i\mu \frac{\partial \alpha}{\partial t_2} + \frac{S}{\varepsilon^2} \alpha + iD_0 \frac{\partial}{\varepsilon} |\alpha + Q_0| \alpha^2 = \frac{\pi \sigma_1 c}{\varepsilon^3} e^{\pi \sigma_1 t_2}, \quad (13)$$

where $\alpha$ is assumed to depend on the slow time variable $t_2 (\equiv \varepsilon^2 t$) and $\mu$, $S (= S_{re} + i S_{im})$, $D_0$ and $Q_0$ are constants given by

$$\mu = 2 + 2 \kappa s_1 \left(1 + \frac{s_1}{\Omega} \right),$$

$$S_{re} = -S_{im} = -\sqrt{\frac{\pi \sigma_1}{2}} \left( \frac{\delta}{\sigma_1} + \kappa \delta \frac{s_1^2}{\Omega} \right).$$

$$D_0 = \frac{128 \kappa \delta s_1^3}{9 \pi \sigma_0^2}, \quad Q_0 = Q_1 + Q_2 + Q_3,$$

with $s_1 = \sigma_0^2/(\sigma_1^2 - \sigma_0^2)$, $\Omega = \sigma_0^2 / \sigma_1^2$, and $\delta = (1/\pi \sigma_0 \sigma_1 \omega_0)^2$, $\omega_0$ being a constant of order unity. For the detailed expressions of $Q_1$, $Q_2$, and $Q_3$, see Eqs. (70) to (72) in Ref. 6.

The experiments measure only the excess pressures in the tube and in the cavities of the resonators, from which the Fourier coefficients are obtained. For the sake of comparison with the experiments, the solutions are rewritten in the form convenient to this purpose. The dimensionless excess pressures $p_{\eta 0} n_0 \phi$ and $p_{\eta 0} n_0 \phi'$ are now renormalized by the equilibrium pressure $p_0$, and their solutions are rearranged in the form of the Fourier series in $t$ with period $2/\sigma_1$ as follows:

$$\frac{p_{\eta 0} n_0}{p_0} \left[ \begin{array}{c} p_1 \\ \vdots \end{array} \right] = \frac{1}{\pi \mu} \left[ S_{re} + Q \left| P \right|^2 \right] \left[ \begin{array}{c} P_{n_0} (x, t_2; e) \\ \vdots \end{array} \right] e^{i \pi \sigma_1 t_2} + c.c. \quad (15)$$

Here note that $p_{\eta 0} n_0 / p_0 = \varepsilon \gamma$ with $\gamma n_0 = \eta n_0 / p_0$, and $P_{n_0}$ and $P_{n_0}$ denote the complex Fourier coefficients which depend not only on $x$ but also on $t_2$ and $\varepsilon$. The Fourier coefficients are related to the coefficients of the asymptotic expansion as follows:

$$P_{n_0} = P_{e_0} = \varepsilon^2 \gamma \left[ \frac{1}{2} \left( 1 - \frac{1}{\sigma_1^2} \right) \left( \frac{1}{\sigma_1^2} \right) \cos \left( 2 \pi (x - 1) \right) \right] |\alpha|^2,$$

$$+ \left( 1 + \frac{1}{\sigma_1^2} \right) \left( \frac{1}{\sigma_1^2} \right) \cos \left( 2 \pi (x - 1) \right) |\alpha|^2,$$

$$P_{e_0} = \varepsilon^2 \gamma \left[ \frac{1}{2} \left( 1 - \frac{1}{\sigma_1^2} \right) \left( \frac{1}{\sigma_1^2} \right) \cos \left( 2 \pi (x - 1) \right) \right] |\alpha|^2,$$

$$+ \left( 1 + \frac{1}{\sigma_1^2} \right) \left( \frac{1}{\sigma_1^2} \right) \cos \left( 2 \pi (x - 1) \right) |\alpha|^2,$$
\[ P = |P_{\text{peak}} e^{i(\pi \Delta \sigma_{\text{peak}} - \pi/2)}| \]  

[see Eqs. (86) and (87) in Ref. 6 in the case without the jet loss]. When the piston is oscillating at the frequency \( \sigma + \Delta \sigma_{\text{peak}} \), the phase in the pressure at \( x=1 \) lags behind the piston displacement just by \( \pi/2 \), while the pressure at \( x=0 \) leads the piston displacement by \( \pi/2 \). The latter implies that the pressure on the piston surface is just in phase with the piston velocity so that the maximum power may be input into the air column at a given displacement amplitude of the piston.

**III. EXPERIMENTAL SETUP**

A tube, resonators, and a driver unit are designed carefully so as to meet conditions required by the theory. Especially the higher harmonic resonances and evanescences are avoided as much as possible. In order to generate high pressure amplitude, it is conjectured that the response curve should become symmetric with respect to the peak just as in the linear response. For this, the value of the coefficient \( Q \), responsible for bending of the response curve, should be made as small as possible. This is substantially to suppress the nonlinear effects including the jet loss. The maximum pressure in the tube is set to be about 10\% of the equilibrium pressure because the weakly nonlinear theory tends to break down as the pressure increases further. The experimental setup used is illustrated in Fig. 2 and described in the following.

**A. Tube and resonators**

A stainless-steel tube of inner diameter 37 mm, length 923 mm, and thickness 9 mm is used. The inner surface is polished so smoothly by honing with surface roughness \( R_z \) (ISO4287) below 0.6 \( \mu \)m as to meet the assumptions used in the theory for the boundary layer. One end of the tube is closed by a flat plate while the other end is connected to the driver unit. The resonator consists of the spherical cavity and the circular throat, both being made of the stainless steel. The volume of the resonator is 14.7 cm\(^3\), while the throat is of inner diameter 7.1 mm, length 72.3 mm, and volume 2.87 cm\(^3\). The natural frequency of the resonator is calculated to be 325 Hz at temperature 26.1 °C. For reference, the natural frequency of the tube without the array takes 188 Hz at the same temperature.

Five resonators are connected to the tube equidistantly with the axial spacing \( d=185 \) mm. The resonator nearest to the closed end at \( x=1 \) is placed at half the axial spacing \( d/2 \) away from the end. Such an arrangement is suggested by the fact that the boundary condition at the closed end may be replaced by the mirror image.\(^7\) In fact, the axial spacing appears to be uniform if the mirror image is taken. The resonator is connected to the tube through the hole in the tube wall with an attachment (see Fig. 2), which secures flush mounting with the inner surface of the tube. The end surface of the attachment has the same radius of curvature as the tube so that the inner surface of the tube may become smooth everywhere including where the resonators are connected.

**B. Driver unit**

Although the plane piston is assumed in the theory, it is difficult to seal the air column tightly in reality because no leak of the air through a narrow gap between the piston head and the tube would be guaranteed. Newly devised is a rubber diaphragm of thickness 2 mm sandwiched by a couple of circular plates. The diaphragm is stretched to cover the whole cross section of the tube and clamped at its edge. Two stainless-steel plates of diameter 27 mm and of thickness 0.8 mm each are bolted together at the center as shown in Fig. 2 to the axis of the linear motor.

When the couple of the plates (called simply a bottom plate hereafter) are displaced from the equilibrium position, the diaphragm stretches to form a frustum with the plate. Assuming the lateral surface to be conical, the volume of the frustum is proportional to the displacement of the bottom plate. The ratio of this volume to the cross-sectional area of the tube gives equivalently the displacement of the plane piston \( x_p \) in the theory. Denoting the displacement of the bottom plate from the equilibrium position by \( x_b \), this is related to \( x_p \) as in the case of the bellows.\(^3\) Taking the greater diameter \( D_g \) as the tube diameter and the smaller one \( D_s \) as
the diameter of the bottom plate, the piston displacement $x_p$ is given equivalently by $x_p = (1 + \chi + \chi^2)x_0/3$ with $\chi = D_j/D_v$. For $\chi = 27 \text{ mm}/37 \text{ mm} = 0.730$, $x_p = 0.754x_0$ and the coefficient of proportion is smaller than unity, whereas it takes the value 1.422 in the case of the bellows.

The bottom plate is driven by a linear motor of voice-coil type with the maximum thrust 72 N (Sumitomo Special Metals: type LV25). The motor reciprocates along its axis and its motion is transmitted to the bottom plate by the rod connecting them. This type linear motor can accurately be controlled by varying the input current. The ac power supply (Takasago: AA2000XG) is used to generate sinusoidal currents. The maximum relative errors in the frequency and the wave form from a purely monochromatic wave are less than $5 \times 10^{-3}\%$ and $0.3\%$, respectively. The magnitude of higher harmonics involved in the displacement of the bottom plate is less than $0.2\%$ of the fundamental one.

### C. Measuring instruments

The displacement of the bottom plate is measured by a laser displacement sensor (Keyence: LB-62) with resolution 50 $\mu$m up to 3 kHz. Excess pressure in the tube or in the cavity of the resonator is measured by two condenser microphones (Brüel & Kjær: type 4941) with sensitivity 0.085 mV/Pa. The microphones are set flush with the inner surface of the tube. When the pressure in the cavity is measured, the microphone is set at the bottom of the cavity (at a position opposite to the throat).

The nonoscillatory (zeroth harmonics or dc) component is measured by two pressure sensors of strain-gauge type (Kyowa: PGM-05KG) with a natural frequency of 3 kHz because the microphones fail to measure it. They are also mounted flush with the flat plate and the tube wall as well as the microphones. The data are processed by the FFT analyzer (Ono Sokki: CF-3600) with maximum number of sampling points 16384 to obtain Fourier coefficients.

The temperature of the air column and the tube wall are measured by two sheath-thermocouples of $K$-type and of diameter 0.3 mm. Both thermocouples are positioned at $x = 0.25$. One is inserted hermetically into the center axis of the tube, while the other is put on the outer surface of the tube wall where the temperature of the wall may be regarded as the equilibrium temperature of the air in the tube. The atmospheric pressure is taken as the equilibrium one in the tube so the ambient pressure is measured by an aneroid barometer.

### D. Evaluation of the parameters

For the tube and the resonators chosen, the various parameters are evaluated. The size parameter $\kappa$ takes the value 0.0737, which is much smaller than the one 0.198 in the previous experiments. Here it is to be noted that $\kappa$ is evaluated by using $V_0$ instead of $V$. The dimensionless angular frequencies $\sigma_0$ and $\sigma_1$, which are independent of the temperature and dependent of $\kappa$ only, take the fixed values at 1.73 and 0.951, respectively. The array makes the natural frequency of the tube lower by about 5% than the one in the tube without the array. Given the value of $\kappa$, no harmonic resonances with the combinations of $m (=0,1,2,\ldots,8)$ and $n (=2,3,4)$ occur for the value of $\sigma_0$ in the range $|\sigma_0 - 1.73| < 0.05$. For the range $|\sigma_0 - 1.73| < 0.01$, $m$ and $n$ may be taken up to 19 and 10, respectively. The higher harmonic evanescent waves are avoided in the present case.

The values of $\delta$ and $\delta_0$ depend on the temperature through $a_0$, $\nu$, and $Pr$ (see Table I and Ref. 3). While the value of $\gamma$ is taken to be a constant 1.40, the dependence of the sound speed on the temperature is taken into account according to $a_0 = 331.5 + 0.61T_0$ m/s for $T_0$ measured in degrees Celsius. At 26.1 $^\circ$C, for example, $\delta$ and $\delta_0$ takes the values 0.0325 and 0.116, respectively, which is to be compared with the previous ones $\delta_0 = 0.0282$ and $\delta_0 = 0.223$. The value of $\delta_0$ is reduced but larger than the one assumed. The parameters in the amplitude equation except for $S$ are evaluated by the geometry only as follows: $\mu = 2.30$, $D = 0.478$, and $Q = -0.735$. For the value of $S_{im} = -S_{re}$, see Table I. The values of $\mu$, $S$, and $D$ are close to the ones in the previous experiments ($\mu = 2.43$, $S_{im} = 0.0430$, $D = 0.398$, and $Q = 15.2$), but the one of $Q$ is dramatically reduced. Thus we may say that the present tube and resonators are designed so that the nonlinear effects except for the jet loss are suppressed significantly at the cost of slight increase in the values of $\delta$ and $D$.

### IV. EXPERIMENTS

The experiments are made with the tube hung vertically and the driver unit on top. The excess pressure in the tube is measured at the position $x = 0.17$, 0.38, 0.50, 0.63, 0.83, and 1, while the one in the cavities of the five resonators at $x = 0.10$, 0.30, 0.50, 0.70, and 0.90. At the same time, the complex Fourier coefficients are obtained. Measurements are made when the steady state of oscillations appear to be achieved at a frequency of the peak in the response. The peak is identified by observing that the phase in the displacement of the bottom plate is ahead of the one of the pressure at $x$.
= 1 just by \( \pi/2 \). Since only two microphones are available, measurements are repeated by changing the positions.

Here it should be remarked that the Fourier coefficients measured are different from the ones in Eq. (15). The series (15) is the expansion with period \( 2/\sigma_1 \), while the coefficients to be measured correspond to the ones in the expansion with period \( 2/(\sigma_1 + \Delta \sigma) \). Therefore it must be rewritten for comparison. In Eq. (15), the factors \( e^{i\pi \Delta \sigma t} (= e^{i\pi \sigma' t_2}) \) are taken out of \( P_n \) and \( P_{cn} \) to form \( e^{i\pi \sigma' t_2} \) as follows:

\[
\begin{align*}
\begin{bmatrix}
P_n(x, t; \varepsilon) \\
P_{cn}(x, t; \varepsilon)
\end{bmatrix}
&= e^{i\pi \sigma' t_2} \begin{bmatrix}
P_n(x, t; \varepsilon) e^{i\pi \sigma' t_2} \\
P_{cn}(x, t; \varepsilon) e^{-i\pi \sigma' t_2}
\end{bmatrix} \\
&\times e^{i\pi \sigma(1 + \sigma' t_2)}, \quad (n = 0, 1, 2, \ldots)
\end{align*}
\]

(26)

The factor \( e^{i\pi \sigma' t_2} \) cancels indeed with the one involved in \( P_n \) and \( P_{cn} \). Setting \( P_n \) and \( P_{cn} \) to be in the following form

\[
P_n = |P_n|^2 e^{i\theta_n}, \quad P_{cn} = |P_{cn}|^2 e^{i\theta_{cn}},
\]

(27)

the complex coefficients on the right-hand side of Eq. (26) are expressed as

\[
P_n e^{-i\pi \sigma' t_2} = |P_n| e^{i\psi_n}, \quad P_{cn} e^{i\pi \sigma' t_2} = |P_{cn}| e^{i\psi_{cn}},
\]

(28)

with \( \psi_n = \theta_n - n\pi \sigma \Delta \sigma \) and \( \psi_{cn} = \theta_n - n\pi \sigma \sigma_{\text{peak}} t \). Using Eq. (25) and the explicit form of \( P_n \) and \( P_{cn} \) in terms of \( P \), \( \psi_n \) and \( \psi_{cn} \) are found to be independent of \( t \).

### A. Pressure profiles and Fourier coefficients

Figure 3 shows the temporal profile of the oscillatory component in the pressure measured on the flat plate at the closed end \( x = 1 \) for the displacement amplitude of the bottom plate \( X_b = 1.39 \) mm where \( \tilde{p}' \) denotes the dimensional excess pressure \( \Delta p' \) minus the oscillatory component corresponding to \( P_1(1) \), and the broken line represents the solution taken up to the second harmonics inclusive, i.e., \( P_1 \) in Eq. (17) and \( P_2 \) in Eq. (18) with the replacement of \( \alpha \) by Eq. (20). It is obvious that the profiles are smooth without any shocks.

For the profile in Fig. 3, Fig. 4 shows how the magnitude of each Fourier coefficient \( |P_n| \) decays as \( n \) increases up to 25th. The triangles indicate the data relative to \( |P_1| \) (=0.101) for the profile in Fig. 3, while the circles indicate, for reference, the decay of the coefficients for a shocked profile measured in the tube without connecting the array. As \( n \) increases, the coefficients of the shock-free profile decay very rapidly at rate between \( n^{-3} \) and \( n^{-2} \) in contrast to the slow decay as \( n^{-1} \) for the shocked profile. As will also be seen in Fig. 10, it is found that the magnitude of the second harmonics involved is about 4% relative to \( |P_1| \) and the profile in Fig. 3 is very close to a monochromatic wave in spite of the presence of the nonlinearity. But a slight discrepancy between the profiles measured and predicted is considered to result from higher harmonics.

### B. Frequency response

In the previous experiments, the frequency response is obtained by measuring the maximum pressure \( \Delta p \) at the closed end against the frequency of excitation. In the present experiments, it is obtained from the first Fourier coefficient measured at the closed end. The magnitude of \( |P_1(1)| \) is plotted against the dimensional frequency of excitation in Fig. 5 by the open triangles, closed triangles, open circles, and closed circles for four values of \( X_b = 0.25, 0.50, 1.00, \) and \( 1.75 \) mm, respectively. It is seen that the data measured fall perfectly on the theoretical curves (22) shown in the solid lines, except for the data designated by the closed circles in the vicinity of the peak. The agreements in the magnitude and the frequency are excellent. Since the value of \( Q \) is small, the curves are symmetric with the peak, as expected.

Next we examine the relation between the peak pressure \( |P_{\text{peak}}| \) and the displacement amplitude of the piston \( c (= X_p/a) \) equivalent to the one of the bottom plate. Figure 6 plots \( |P_{\text{peak}}| \) vs \( c \) by the circles in the log–log scales. It is already shown in Ref. 6 that for \( \Gamma \ll S_m^2/4D \), \( |P_{\text{peak}}| \) is given by \( \Gamma/S_m \) while for \( \Gamma \gg S_m^2/4D \), \( |P_{\text{peak}}| \) is given by \( \sqrt{\Gamma/D} \). Thus while \( \Gamma \), i.e., \( c \) is small, the peak pressure is proportional to \( c \), but it becomes proportional to \( c^{0.5} \) as \( c \) becomes.
larger. The changeover is defined to occur at $\Gamma=S_0^2/4D$, which gives $\Gamma=1.33 \times 10^{-3}$ and $c=1.59 \times 10^{-4}$. The data measured lie on the line $c^{0.77}$ approximately, which is located between two lines $c$ and $c^{0.5}$. The data seem to continue on the line $c^{0.77}$ beyond $c=10^{-3}$.

C. Axial distributions of the first harmonics

Next the pressure field in the tube and in the cavities of the resonator is measured and checked against the theory. Figures 7 and 8 show, respectively, the axial distributions of the first Fourier coefficients $|P_1|\,e^{i\phi_1}$ and $|P_{c1}|\,e^{i\phi_{c1}}$ where (a) and (b) represent, respectively, each magnitude and phase in degree. Here the open triangle, closed triangle, open circle, and closed circles represent, respectively, the data for $X_b =0.52, 0.82, 1.09$, and $1.33$ mm, respectively. The solid lines represent the theoretical distributions calculated by $P_1$ and $P_{c1}$ in Eq. (17) with the replacement of $\alpha$ by Eq. (20). The change in sign of $P_1$ and $P_{c1}$ with respect to $x$ is taken into the respective phases $\phi_1$ and $\phi_{c1}$.

The measured data fall on the curves predicted by the theory. Quantitatively good agreements are seen not only in the tube but also in the cavities. Note that the maximum excess pressure in the cavity is about 50% greater than the one in the tube. It turns out that the terms of $|P|^3$ yield the corrections of a few percent. Thus the nonlinearity is suppressed to be so small that the pressure distributions may be regarded as being close to the linear solutions. The above-noted agreements support the validity of the assumptions of the one-dimensional field in the tube and the continuum approximation for the resonators. There are only five resonators that are connected to the tube.

D. Axial distributions of the zeroth and second harmonics

We proceed to check the zeroth (dc) and second harmonic components in the pressure field. At first, the zeroth components $P_0$ and $P_{c0}$ are measured. Figure 9 shows the axial distributions of $\Delta P_0$ for the deviation of $P_0(x)$ from $P_0(1)$. The symbols in figures correspond to the amplitudes of the bottom plate specified in Figs. 7 and 8, while the solid lines represent the distributions calculated by $P_0$ in Eq. (16) with Eq. (20). Here it should be remarked that $P_0$ tends to increase slowly in the course of time because the mean temperature and pressure in the tube increase slowly due to heating by friction on the tube wall. Thus the deviation $\Delta P_0$ is plotted in Fig. 9. The zeroth harmonics is seen to agree well with the theory.

Figures 10 and 11 show, respectively, the axial distributions of the second Fourier coefficients $|P_2|\,e^{i\phi_2}$ and $|P_{c2}|\,e^{i\phi_{c2}}$ in the tube and in the cavities where (a) and (b) represent each magnitude and phase in degree. As two nodes are pre-
the data measured for the displacement amplitude of the bottom plate.

The open triangles, closed triangles, open circles, and closed circles indicate respectively, the distributions of the magnitude $|P_1|$ and the phase $\phi_1$ in degree; the open triangles, closed triangles, open circles, and closed circles indicate the data measured for the displacement amplitude of the bottom plate $X_b = 0.52$, 0.82, 1.09, and 1.33 mm, respectively; the solid lines show the theoretical distributions calculated by $P_1$ in Eqs. (17) with (20).

Predicted near $x = 0.25$ and 0.75, the data measured show the axial distributions similar to the theoretical ones calculated by $P_2$ and $P_{c2}$ in Eq. (18) with Eq. (20). While the amplitude of the bottom plate is small, the data for the magnitude are close to the curve predicted. Generally speaking, however, the data measured are greater than the ones by the theory in the middle of the tube and smaller near both ends. For the second harmonics as well, the magnitude in the cavity is remarkably larger than the one in the tube.

But there is a significant discrepancy in the phase. From

FIG. 9. Axial distributions of the deviation of the zeroth Fourier coefficient $|P_0|e^{i\phi_1}$ for the pressure in the cavity of the resonator where (a) and (b) represent, respectively, the distributions of the magnitude $|P_0|$ and the phase $\phi_1$ in degree; the open triangles, closed triangles, open circles, and closed circles indicate the data measured for the displacement amplitude of the bottom plate $X_b = 0.52$, 0.82, 1.09, and 1.33 mm, respectively; the solid lines show the theoretical distributions calculated by $P_0$ in Eqs. (16) with (20).

The second harmonics of $e^2$ [see Eqs. (54) and (55) in Ref. 6], it is found that $\psi_2$ and $\psi_{2c}$ take the values 0 or $\pi$ depending on the sign. Because the frequency of the second harmonics exceeds the resonance frequency of the Helmholtz resonator, $\psi_{2c}$ is different from $\psi_2$ by $\pi$. But it is seen that the data measured scatter in between and the phases differ by multiples of $45^\circ$.

V. DISCUSSIONS

In designing the tube and the resonators, higher harmonic resonances and evanescences have been avoided as much as possible. In consequence, higher harmonics are suppressed significantly. It is seen from Eq. (3), however, that as $m$ increases, $\sigma_m^+$ approach $m$, while $\sigma_m$ approach $\sigma_0$, so that the effect of dispersion tends to disappear as $m$ increases. Thus it becomes difficult to avoid the harmonic resonances for even higher modes. It would eventually determine the upper limit of the peak pressure how far the state of out-of-resonance can be achieved. Therefore it is essential to postpone the resonance as far as possible in order to yield a higher peak in the frequency response.

All theoretical results used for comparison are calculated by using the size parameter $\kappa$ and the natural frequency of the resonator $\omega_0$ based on the total volume of the resonator $V$, instead of the volume of the cavity $V_c$. If those values by the original definitions are used in the frequency response for example, then the peak values are higher by several percent while each curve is shifted upward by a few hertz. In this
case, it is hard to acknowledge the perfect agreements. This is why \( V \) has been used in the comparisons that follow. It implies that when the volume of the throat is not negligible, the equation for the conservation of mass in the whole resonator would be more appropriate than the one in the cavity. But it is remarked that if the throat is included, then the numerator would be more appropriate than the one in the cavity.

Next we consider physical origin of the discrepancy of the second Fourier coefficients. It may be attributed to the fact that the value of \( \delta_{i} (=0.116) \) for the throat friction takes a larger value than assumed. In the present experiments, most concerned is the reduction of the value of \( Q \) whereas the value of \( \delta_{i} \) is less concerned because its effect on the parameter \( S \) in the amplitude equation is weakened by the factor \( \kappa \). As the wall friction becomes large, the magnitude of the pressure field is expected to be suppressed in general. Strange enough, however, the pressure in the tube and the cavities measured at the positions in the middle of the tube is higher than the one predicted. The wall friction does not resolve the discrepancy in the magnitude, but it may be promising to explain a phase difference in the multiples of 45°. Because the effect of the wall friction in Eq. (95) in Ref. 6 is expressed in the form of the derivative of three-"order with respect to the time, the factor \( \Gamma^{3/2} \) \( (=e^3/n=1) \) appears, which yields the phase difference by 45°. In fact, a tendency for this deviation is seen in Figs. 9(b) and 10(b) but is not conclusive yet.

The origin may also be attributed to the jet loss taken in the semi-empirical form. The present model does not take account of asymmetry in the jet loss with respect to the direction of flow in the throat. Physically there should be some difference between the jet loss when the air flows from the tube into the small cavity and the one when it flows from the cavity into the tube. Such a difference has not been considered. Imagine that the value of the coefficient of the jet loss depends on the direction of the flow. Then the second harmonics would appear in the solutions of order \( \epsilon^2 \) in addition to the odd harmonics for the symmetric jet loss [see Eq. (97) in Ref. 6]. Since the measured data correspond to the solutions including all order in \( \epsilon \), it is likely that the second harmonics due to the asymmetry might contaminate the ones of order \( \epsilon \) predicted by the theory. But this is not beyond the speculation as well.

The theoretical framework on the basis of the assumptions (4) and \( \delta_{i}=O(\epsilon^2) \) is typical and essential to derivation of the amplitude equation (13). If this framework were modified or changed, a different but less typical situation would appear for the resonant excitation. In order to see better agreements of the second coefficient in the present framework, reduction of the values of \( \delta_{i} \) and of \( D \) for the jet loss is necessary even at the cost of increase of the parameter value of \( Q \).

VI. CONCLUSIONS

The validity of the weakly nonlinear theory for high-amplitude and shock-free oscillations of the air column in the tube with the array of Helmholtz resonators has been checked against the data measured in the experiments. By avoiding the harmonic resonances and evanescences and reducing the value of \( Q \) in the amplitude equation, higher harmonics have been suppressed significantly in spite of the presence of nonlinearity. In consequence, the pressure profile appears to be nearly sinusoidal and the curves of the frequency response become symmetric with respect to the peak without any bending just as in the linear case. It has resulted in the quantitatively good agreements with the one predicted by the theory up to the peak pressure of about 170 dB (SPL). In order to achieve shock-free, high-amplitude oscillations, the condition for a tube to be dissonant is necessary but not sufficient. On top of dissonance, it is crucial to guarantee how far harmonic resonances and evanescences can be avoided.

The agreements of the first Fourier coefficients for the pressure field in the tube and in the cavities endorse the validity of the underlying assumptions of the one-dimensional field averaged over the cross section of the main flow and the continuum approximation for the array of resonators. As the slight discrepancy in the frequency response occurs at the peak pressure of about 170 dB (SPL), the weakly nonlinear theory tends to break down near this level and beyond it. Perhaps effects of the acoustic streaming would be pronounced and the flow field would no longer be almost one-dimensional. In addition, the formation of the high-speed jets from the orifices would make the field very complicated. It is unknown yet whether or not these would
be responsible for the quantitative discrepancy in the second Fourier coefficients. In view of the good agreements of the frequency response, however, it may be concluded that the theory is valid and useful enough to provide guidelines in designing accurately the array of Helmholtz resonators up to the present level of excitation.

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