Localized oscillations of a spatially periodic and articulated structure

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Abstract

Localized oscillations are examined in flexural motions of a spatially periodic and articulated structure of uniform and rigid members connected with neighboring ones through a coupler giving nonlinear restoring moment. Formulation is made in terms of the Lagrangian with holonomic constraints for continuity of displacements at the junctions, and is also given in the form of constrained Hamiltonian system. Nonlinearity results from not only material response due to anharmonic potential in the restoring moment (linear plus hard, cubic spring) but also geometrically finite displacements, the latter of which induces longitudinal motions to couple, in turn, with the flexural (transverse) motions. In particular, the axial tension introduces nonlocal moment for rotation of each member in addition to the one due to nearest neighbors. Numerical calculations and asymptotic analysis show existence of time-periodic localized oscillations while the amplitude remains small. As it becomes larger, however, quasiperiodic oscillations emerge, and there also occur such cases that they are not decayed temporarily but are delocalized.

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1. Introduction

Since localized oscillations now known as the intrinsic localized mode (ILM) or discrete breather (DB) were discovered in the late 80s by Sievers and Takeno [1], many investigations have been made in the context of lattice dynamics, and some review articles are now available [2–5]. Basically they mean spatially localized and time-periodic oscillations, which occur stably and stationary in nonlinear, spatially discrete, and perfectly periodic systems without any defects and dissipations. Although initial models are concerned with one-dimensional lattice (nonlinear, classical Hamiltonian lattice), studies now extend to multi-dimensional systems, movable ILMs, and forced (and dissipative) systems. In this special issue, the first two topics are reviewed.
and considered by Feng and Kawahara [6], and Yoshimura and Doi [7], respectively, and the last topic was studied by Martínez et al. [8], Sato et al. [5] and Maniadis and Bountis [9].

Although the initial studies on ILM (or DB) are theoretical ones in the context of solid-state physics or nonlinear wave dynamics, the concept now tends to spread into various fields such as chemistry, optics, biophysics and so on, with experimental evidence [4]. In view of this, the concept will be, surely enough, important and indispensable in designing mechanical systems such as not only micro-electro-mechanical systems (MEMS) [5] but also large-scaled space and marine structures like “Mega-Float” [10]. In fact, they are often made by combination of a number of identical units, for example, a multi-span beam or panel, a periodic truss or a periodically ribbed panel. In these systems, the localization due to irregularity or disorder has been investigated in detail [11,12], but to authors’ knowledge, ILM (or DB) seems to be unknown yet.

In such structures, it is to be noted, at first, that flexural (transverse) motions are of primary importance rather than longitudinal ones as in lattice dynamics, and both will couple with each other if finite geometrical displacements are taken into account. This geometrical nonlinearity is expected to introduce new features in addition to the material nonlinearity due to anharmonic potential. This paper concerns, as a simplest example, with a periodic and articulated structure made of identical units of beams or panels, as shown in Fig. 1(a). No real structures extend infinitely of course so the term “periodic” may be inappropriate in exact sense. But as the localization implies, it is essential for only a finite number of units among many ones to be engaged in the oscillations, though boundary conditions at both ends are crucial even in the purely periodic case.

In the articulated structures, “weak points” are at junctions, which are bent easier than each unit. When the junctions exhibit resilience against rotation or have stoppers or bumpers, their effect may be taken into account in the form of a coupler as modeled in Fig. 1(b), which yields nonlinear restoring moment usually by hard springs. If a “rotational modulus” $K$ in the restoring moment is smaller than the bending rigidity $EI$ of the unit divided by a typical length $l$ of the unit, i.e. $K \ll EI/l$, then the unit may be regarded as being rigid. Unless the condition $EI/ Kl \equiv \chi \gg 1$ is satisfied, elastic response of each unit must be taken into account. Then there appear a variety of phenomena due to reflections and transmissions of flexural waves at the junctions so that the banded structure emerges in the dispersion relation [13]. The present model will be valid for slow oscillations whose frequency is far below the natural frequency of each beam or panel if elastic (see Fig. 9 in [13] in the case with $\chi = 100$ and $\Omega \ll 1$, where the wide stopping band prevails above the low-frequency passing band).

One of the characteristics of ILMs (or DBs) in the lattices subjected to force due to nearest neighbors is that while the amplitude of oscillations decays exponentially in space, they are time-periodic with a single frequency [2]. On the other hand, quasiperiodic localized oscillations having two incommensurate frequencies are shown to exist as exact solutions by Cai et al. [14] for the Ablowitz–Ladik lattice (discretization of nonlinear Schrödinger equation), but they are regarded as being nongeneric and exceptional. Bambusi [15] showed that quasiperiodic oscillations, though not exact solutions, are good approximation to a real trajectory for a long time. Recently, however, Gorbach and Flach [16] showed existence of quasiperiodic breathers in nonlinear and nonlocal dispersive systems subjected to long-range force. In the present articulated structure,
nonlocal moment appears through the tension due to longitudinal motions, and quasiperiodic localized oscillations emerge.

In what follows, model and formulation for the articulated structure are presented in Section 2, in which two sets of equations are derived, one being obtained by eliminating the holonomic constraints at each junction and the other presented in the Hamiltonian form with constraints. In Section 3, numerical solutions to the equations are sought by applying the standard and symplectic Runge–Kutta methods and discussions on the results are given in Section 4.

2. Model and formulation

As is shown in Fig. 2, the structure consists of \( N (\geq 2) \) identical, long and rigid members, which are of length \( l \) and of uniform density \( \rho \) per unit axial length, and connected to adjoining ones by a coupler giving nonlinear restoring moment. Each member may be a beam or a panel, and the mass of the coupler and the interval of the junction are negligible. A member with couplers on both sides forms one unit of the structure.

Supposing that the number of the units \( N \) is large but finite, they are numbered consecutively by integer \( j \), and physical variables in the unit \( j \) are denoted by attaching subscript \( j \). The junction located at the left end of the \( j \)th unit is called the \( j \)th junction.

Motions of the structure are restricted in the \((x, y)\) plane where the \( x \)-axis is taken along the structure in equilibrium. In the \( j \)th unit, the position of the center of mass of the unit is denoted by \((x_j(t), y_j(t))\) and the angle of the centerline to the \( x \)-axis is denoted by \( \phi_j(t) \), \( t \) being the time. Suppose that the restoring moment (torque) \( M_j \), which is affected by the coupler at the left end of the \( j \)th unit, is given by a linear plus cubic function of difference in angle between two centerlines of the adjacent units in the following form:

\[
M_j = K(\phi_j - \phi_{j-1}) + K_C(\phi_j - \phi_{j-1})^3,
\]

for \( 2 \leq j \leq N \), \( K \) and \( K_C \) being constant.

2.1. Euler–Lagrange equations

Equations of motions are easily derived by applying the Newton’s law of motions to each unit [13] or by calculating the Lagrangian of the whole structure. Adopting the latter method, we start with normalizing the variables by replacing them as follows: \( x_j \rightarrow lx_j, y_j \rightarrow ly_j \) and \( t \rightarrow (\rho l^3/K)^{1/2}t \).

The Lagrangian of the system \( L \) is easily calculated to be

\[
L = \sum_{j=1}^{N} \left[ \frac{1}{2} \dot{x}_j^2 + \frac{1}{2} \dot{y}_j^2 + \frac{1}{2} I \dot{\phi}_j^2 \right] - \sum_{j=1}^{N-1} \left[ \frac{1}{2} (\phi_{j+1} - \phi_j)^2 + \frac{1}{4} \kappa (\phi_{j+1} - \phi_j)^4 \right],
\]

where the dot designates differentiation with respect to \( t \), \( I (= l/12) \) denotes the dimensionless moment of inertia of each member about the center of mass, and \( \kappa = K_C/K \). Since the Lagrangian does not take account of existence of the junctions, geometrical constraints are necessary for continuity of displacement, which are given for the \( j \)th junction (\( 2 \leq j \leq N \)) as

Fig. 2. Configuration of the articulated structure where the position of the center of mass in the \( j \)th unit is denoted by \((x_j, y_j)\) and the angle of centerline of the unit to the \( x \)-axis is denoted by \( \phi_j \).
Both ends of the structure are assumed to be free. Such holonomic constraints are taken into account by introducing Lagrange multipliers $\lambda_j(t)$ and $\mu_j(t)$ ($2 \leq j \leq N$) to redefine a Lagrangian $\mathcal{L}$ by

$$\mathcal{L} = L + \sum_{j=2}^{N} (\lambda_j f_j + \mu_j g_j).$$

Taking generalized coordinates $\xi_n$ with integer $n$ ($1 \leq n \leq 3N$) as $\xi_j = x_j, \xi_{N+j} = y_j$ and $\xi_{2N+j} = \phi_j$ ($1 \leq j \leq N$), the Euler–Lagrange equations for the new Lagrangian, $(d/dr)(\partial L/\partial \dot{\xi}_n) - \partial L/\partial \xi_n = 0$, leads to

$$\frac{d}{dr} \left( \frac{\partial L}{\partial \dot{\xi}_n} \right) - \frac{\partial L}{\partial \xi_n} = \sum_{j=2}^{N} \left( \lambda_j \frac{\partial f_j}{\partial \dot{\xi}_n} + \mu_j \frac{\partial g_j}{\partial \dot{\xi}_n} \right).$$

Eq. (5) is reduced to the Newton’s equations for translation and rotation about the center of mass in the $j$th unit ($1 \leq j \leq N$):

$$\frac{d^2 x_j}{dr^2} = \lambda_{j+1} - \lambda_j,$$

$$\frac{d^2 y_j}{dr^2} = \mu_{j+1} - \mu_j,$$

$$\frac{1}{12} \frac{d^2 \phi_j}{dr^2} = T_j + v_j,$$

with

$$T_j = M_{j+1} - M_j = (\phi_{j+1} - \phi_j) + \kappa (\phi_{j+1} - \phi_j)^3 - (\phi_j - \phi_{j-1}) - \kappa (\phi_j - \phi_{j-1})^3$$

and

$$v_j = -\frac{1}{2}(\lambda_j + \lambda_{j+1}) \sin \phi_j + \frac{1}{2}(\mu_j + \mu_{j+1}) \cos \phi_j,$$

where $M_1, \lambda_1, \mu_1, M_{N+1}, \lambda_{N+1}$ and $\mu_{N+1}$ are taken to vanish formally. It is found from (6) that the Lagrangian multipliers $\lambda_{j+1}$ and $\mu_{j+1}$ correspond physically to the $x$ and $y$ components of the tension, respectively, at the $(j+1)$th junction acting on the $j$th unit, which are directed toward the respective positive directions [13]. On the $(j+1)$th unit, of course, they act in the negative direction of the $x$ and $y$ axes. Eqs. (6) and the holonomic conditions (3) constitute the full set of equations. The Lagrange equations of motion for degree of freedom $3N$ are subjected to $2(N-1)$ holonomic constraints (3).

It is immediately found from (6a) and (6b) that

$$\sum_{j=1}^{N} \frac{d^2 x_j}{dr^2} = 0 \quad \text{and} \quad \sum_{j=1}^{N} \frac{d^2 y_j}{dr^2} = 0.$$  

These relations indicate uniform translation of the structure with a constant velocity. The total energy $E$ defined by

$$E = \sum_{j=1}^{N} \left( \frac{1}{2} \dot{x}_j^2 + \frac{1}{2} \dot{y}_j^2 + \frac{1}{2} \dot{\phi}_j^2 \right) + \sum_{j=1}^{N-1} \left[ \frac{1}{2} (\phi_{j+1} - \phi_j)^2 + \frac{1}{4} \kappa (\phi_{j+1} - \phi_j)^4 \right]$$

is of course conserved, and the angular momentum $P$ defined by

$$P = \sum_{j=1}^{N} \left( x_j \dot{y}_j - \dot{x}_j y_j + I \dot{\phi}_j \right)$$

(8)
is conserved as well.  
In order to solve (6), the Lagrange multipliers may be eliminated by using

$$\lambda_j = \sum_{k=1}^{j-1} \frac{d^2x_k}{dt^2} \quad \text{and} \quad \mu_j = \sum_{k=1}^{j-1} \frac{d^2y_k}{dt^2},$$

(10)
to be substituted into the equations for \( \phi_j \). It then follows that

$$\left( \sum_{k=1}^{j} \frac{d^2x_k}{dt^2} - \frac{1}{2} \frac{d^2x_j}{dt^2} \right) \sin \phi_j - \left( \sum_{k=1}^{j} \frac{d^2y_k}{dt^2} - \frac{1}{2} \frac{d^2y_j}{dt^2} \right) \cos \phi_j + \frac{1}{12} \frac{d^2\phi_j}{dt^2} = T_j,$$

(11)

where the summations vanish for \( j = N \). These equations are complemented by the holonomic constraints differentiated twice with respect to \( t \). Since the number of these conditions is \( 2(N - 1) \), two Eqs. (7) are employed to pose the second-order equations for \( 3N \) unknowns of \( x_j, y_j \) and \( \phi_j (1 \leq j \leq N) \). In passing, \( x_j \) and \( y_j \) may further be eliminated from these equations to derive \( N \) equations for \( \phi_j \), though their explicit expressions are too complicated to be reproduced here. It should be remarked that the tensions \( \lambda_j \) and \( \mu_j \) introduced in (11) the nonlocal moments in the form of sum in addition to the local ones \( T_j \) due to nearest neighbors.

Without eliminating the Lagrange multipliers, alternatively, they may be sought as unknowns as well. Introducing generalized momenta \( \eta_n (1 \leq n \leq 3N) \) by \( \eta_n = \partial L/\partial \dot{\zeta}_n \) as \( \eta_j = p_j, \eta_{N+j} = q_j \) and \( \eta_{2N+j} = \psi_j (1 \leq j \leq N) \), (6) may be rewritten into Hamiltonian form with constraints by taking the Hamiltonian \( H \) as the total energy of the system \( E \). In fact, they are expressed in terms of the generalized variables in the \( j \)th unit \( (1 \leq j \leq N) \) as

$$\frac{d}{dt} \begin{bmatrix} x_j \\ y_j \\ \phi_j \end{bmatrix} = \begin{bmatrix} \partial H/\partial p_j \\ \partial H/\partial q_j \\ \partial H/\partial \psi_j \end{bmatrix} = \begin{bmatrix} p_j \\ q_j \\ 12\psi_j \end{bmatrix},$$

(12a)

$$\frac{d}{dt} \begin{bmatrix} p_j \\ q_j \\ \psi_j \end{bmatrix} = - \begin{bmatrix} \partial H/\partial x_j \\ \partial H/\partial y_j \\ \partial H/\partial \phi_j \end{bmatrix} + \begin{bmatrix} \partial h/\partial x_j \\ \partial h/\partial y_j \\ \partial h/\partial \phi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ T_j \end{bmatrix} + \begin{bmatrix} \lambda_{j+1} - \lambda_j \\ \mu_{j+1} - \mu_j \\ \nu_j \end{bmatrix},$$

(12b)

with \( h \) defined by

$$h = \sum_{j=2}^{N} (\lambda_j f_j + \mu_j g_j),$$

(12c)

together with (3). These constitute the differential-algebraic equations where the Hamiltonian \( H=E(H(\zeta_n, \eta_n) = E) \) is constant along solutions because

$$\frac{\partial H}{\partial t} = \sum_{n=1}^{3N} \left( \frac{\partial H}{\partial \dot{\zeta}_n} \dot{\zeta}_n + \frac{\partial H}{\partial \dot{\eta}_n} \dot{\eta}_n \right) = \sum_{n=1}^{3N} \left[ \frac{\partial H}{\partial \zeta_n} \frac{\partial H}{\partial \eta_n} + \frac{\partial H}{\partial \eta_n} \left( - \frac{\partial H}{\partial \zeta_n} + \frac{\partial H}{\partial \zeta_n} \right) \right] = 0,$$

(13a)

with

$$\sum_{n=1}^{3N} \frac{\partial f_j}{\partial \zeta_n} \dot{\zeta}_n = \sum_{n=1}^{3N} \frac{\partial g_j}{\partial \zeta_n} \dot{\zeta}_n = 0 \quad (2 \leq j \leq N).$$

(13b)

Here it is noted that the original system may also be formulated in the Hamiltonian form by using (3) to eliminate \( 2(N-1) \) generalized coordinates. In this formulation, the Hamiltonian takes a very complicated form, if \( N \) is large, and the kinetic energy contains \( \zeta_n \) as well as \( \eta_n \).

2.2. Linear dispersion relation

Here we examine linear wave propagation along the structure by assuming \( N \) to be infinite [13]. Supposing that \( |\phi_j| \ll 1 \), (3a) are approximated to be \( x_j - x_{j-1} = 1 \). This implies together with (6a) that the \( x \) coordinate...
of the center of mass in each unit is immobile and therefore all $\lambda_j$ vanish, namely, the tension in the $x$ direction vanishes at every junction. The other constraints (3b) and the linearized equations of (6b) and (6c) lead to the following ones:

$$y_{j-1} + \frac{1}{2} \phi_{j-1} = y_j - \frac{1}{2} \phi_j,$$

$$\frac{d^2 y_j}{dt^2} = \mu_{j+1} - \mu_j,$$  \hspace{1cm} (14a)

$$\frac{1}{12} \frac{d^2 \phi_j}{dt^2} = (\phi_{j+1} - \phi_j) - (\phi_j - \phi_{j-1}) + \frac{1}{2} (\mu_{j+1} + \mu_j).$$  \hspace{1cm} (14b)

Eliminating $\phi_j$ and $\mu_j$, (14) are reduced to difference and differential equations for $y_j$ as follows:

$$\frac{1}{2} \frac{d^2}{dt^2} (y_{j+1} + 2y_j + y_{j-1}) + 2(y_{j+2} - 4y_{j+1} + 6y_j - 4y_{j-1} + y_{j-2}) - \frac{1}{6} \frac{d^2}{dt^2} (y_{j+1} - 2y_j + y_{j-1}) = 0.$$  \hspace{1cm} (15)

The dispersion relation of (15) is derived by assuming a sinusoidal wave in the form of $y_j \propto \exp[i(kj - \omega t)]$, $k$ and $\omega$ being a wavenumber and an angular frequency, respectively, as follows:

$$\omega^2 = \omega_0^2 \left[ \frac{\sin^4(k/2)}{3 - 2 \sin^2(k/2)} \right],$$  \hspace{1cm} (16)

with $\omega_0 = \sqrt{48} \approx 6.93$. Fig. 3 shows the dispersion relation for $\omega$ versus $k$. Although the curve is repeated periodically with respect to $k$, it is restricted within the range $|k| \leq \pi$. It is found that the cutoff occurs at $\omega = \omega_0$, beyond which propagation is prohibited. At the cutoff angular frequency $\omega_0$, $k$ takes $\pi$, which corresponds to the shortest wavelength twice the unit length, where the structure is zigzagged. This mode is called $\pi$-mode.

3. Numerical analysis

3.1. Initial conditions

Equations of motions derived in the preceding section are solved numerically. As the initial conditions, the values for $\phi_j$ are imposed in the form of the $\pi$-mode whose envelope is modulated in the form of a pulse as

$$\phi_j(0) = (-1)^{j-1} A \sech[x(j-c)],$$  \hspace{1cm} (17)

for $1 \leq j \leq N$ where $A$ and $x$ are arbitrary constants, $c (=N/2 + 1/2)$ being the center of the structure. The initial positions $x_j$ and $y_j$ are determined by (3) if one set of $(x_j, y_j)$ is prescribed. All initial velocities are taken to vanish, $p_j = q_j = \psi_j = 0$ for $1 \leq j \leq N$ so that the center of mass of the system remains fixed at the origin, while it has no angular momentum, i.e. it does not rotate as a whole.

With these initial conditions prescribed, the Lagrange multipliers are obtained from the holonomic constraints. Differentiating (3) twice with respect to $t$ and using (12), it follows that

![Fig. 3. Dispersion relation (16) for $-\pi \leq k \leq \pi$ where no real $\omega$ exists for $\omega > \omega_0 = \sqrt{48}$.](image-url)
The center of mass of the system is located at the junction $k$. In this consequence, the magnitude of the profile is antisymmetric as shown in Fig. 4 (b) by the broken line. But qualitative features of oscillations to be described below are the same as the ones for the symmetric profile.

Fig. 4. Spatial profile of the structure where (a) shows the profile for $N = 64$ and the initial values (17) with $A = \pi/180$ and $\alpha = 0.6$, and (b) shows the symmetric and antisymmetric profiles with the same values of $A$ and $\alpha$ drawn, respectively, for $N = 64$ and $N = 65$ by solid and broken lines.
have recently been made to devise new schemes to solve differential equations for Hamiltonian systems in view of application to molecular dynamics, perhaps independently of the study of ILMs. The point is to preserve certain structures and invariants, in particular, the symplecticness and the conservation of the total energy for a long-time integration (see the latest monograph by Hairer, Lubich and Wanner [17]).

The first-order Eqs. (12) constitute the constrained Hamiltonian system with (3). They are the differential-algebraic equations with index three. Incidentally the index means the number of differentiation to derive differential equations from the algebraic constraints. To authors’ knowledge, the symplectic integrators of fourth or higher order for the constrained Hamiltonian system are few. Only available ones are found in a series of papers by Jay [18–20]. Among them, the symplectic partitioned Runge–Kutta method consisting of three stage Lobatto IIIA and Lobatto IIIB is used in this paper.

3.2. Numerical results

Under the initial conditions mentioned above, the calculations are carried out for various values of $\kappa$ and $\alpha$ but with $A$ fixed at $\pi/180$. Increase in the value of $\kappa$ with $A$ fixed corresponds, alternatively, to choice of a larger value of $A$ if the value of $\kappa$ were fixed. The value of $\alpha$ controls the spatial spread of the initial pulse and the larger the value becomes, the narrower the pulse width.

The typical solutions in the case with $\kappa = 5000$ and $\alpha = 0.6$ are displayed in Figs. 6–8. Fig. 6 shows the spatio-temporal profile of the structure up to $t = 200$. It is seen that the initial profile is neither decayed nor dispersed in the course of time, but is localized around $x = 0$. Fig. 7 shows the spatial profile at $t = 193$ where $\phi_{32}$ takes nearly the maximum. The profile decays exponentially but small oscillations remain because the initial condition differs from the localized solution so that some radiation occurs transiently but it is confined in the finite region. Fig. 8 shows the temporal variations of $\phi_{28}$, $\phi_{29}$, $\phi_{30}$, $\phi_{31}$ and $\phi_{32}$ around $t = 193$. The maximum amplitude of oscillations $|\phi_{32}|_{max}$ is slightly greater than the initial one $\phi_{32}$ at $t = 0$.

Fig. 9 shows the Fourier spectrum of the oscillations of $\phi_{32}$ sampled after the initial transients are almost decayed out at $t = T_0 \approx 35$. Defining the Fourier transform of a discrete time series of $\phi_{32}$ at $t = T_0 + m\Delta t$ ($m = 1, 2, 3, \ldots, M$), denoted by $\tilde{\phi}_{32}(m)$, $\Delta t$ being a small time interval such that $M\Delta t \equiv T$, to be

$$\tilde{\phi}_{32}(l) = \frac{1}{M} \sum_{m=1}^{M} \phi_{32}(m) \exp(-2\pi ilm/M),$$

(19)

($l = 1, 2, 3, \ldots, M$), $\tilde{\phi}_{32}(l)$ is regarded as a function of the angular frequency $\omega (=2\pi m/T)$. The highest peak is located at $\omega = \omega_1 (=12.27)$, and the second highest peak at $\omega_2 (=7.67)$ except for the third harmonics of $\omega_1$. It is found that both frequencies $\omega_1$ and $\omega_2$ are higher than the cutoff angular frequency, and the oscillations are quasiperiodic because the ratio $\omega_2/\omega_1$ is not commensurate in general. In fact, when the orbit in the
phase space is cut by a plane \((\phi_{32}, \psi_{32})\), Fig. 9 shows the Poincaré section which draws a ring with finite band width due to a torus.

Next we show the results in the case where the value of \(j\) is decreased to 1500 for \(a = 0.6\). The Fourier spectrum in Fig. 10 shows that the peaks appear at \(\omega_1 (=8.48)\), \(3\omega_1\) and \(5\omega_1\) only and no other peaks corresponding to \(\omega_2\) in the preceding case appear. In this case, the Poincaré section in Fig. 10 shows a thin ring, which implies the periodic oscillations. It is expected from this that as the magnitude of \(j\) is decreased, the oscillations tend to be periodic.

On the contrary, as the magnitude of \(j\) is increased, it is found that the oscillations are no longer localized around \(x = 0\) but delocalized to spread over the whole system. Fig. 11 shows the spatio-temporal profile of the structure up to \(t = 200\) for the case with \(k = 100,000\) and \(a = 0.6\). As is shown in Fig. 12, the Fourier spectrum of \(\phi_{32}\) has no sharp peaks but a broadband one, while the ring in the Poincaré section \((\phi_{32}, \psi_{32})\) is lost. As the
amplitude of oscillations becomes larger, the quasiperiodic oscillations become unstable to be delocalized and chaotic.

For other combinations of the values of the parameters, Fig. 13 shows the relations of the primary frequency $\omega_1$ and the secondary one $\omega_2$ versus the maximum amplitude of $\phi_{32}$ in the localized oscillations. The existence is judged by the calculations up to $t = 200$. The same symbols indicate the results with the value of $\alpha$ fixed: circle $\bigcirc$ ($\alpha = 0.6$), box $\square$ (0.9), diamond $\blacklozenge$ (1.2) and triangle $\triangle$ (1.5), and the solid and blank symbols designate $\omega_1$ and $\omega_2$ in the respective cases.

It is found that the maximum amplitude of the localized oscillations increases with the value of $\kappa$ but decreases with $\alpha$. For reference, $\omega_0$ and $A$ are drawn by horizontal and vertical lines, respectively. It is also found that $\omega_1$ is always greater than $\omega_0$ and the difference $\omega_1 - \omega_2$ increase with the value of $\kappa$. As this value becomes smaller, the peak at $\omega_2$ becomes invisible so no blank symbols are drawn. Then the oscillations are time-periodic with a single frequency indicated by the solid symbol. In summary, it is found that while the value of $\kappa$ is small, the oscillations are periodic, but as it becomes larger, the quasiperiodic oscillations bifurcate to emerge and further they tend to be delocalized to spread over the whole structure.
At first, we discuss the limiting behavior of the localized solutions as the frequency approaches the cutoff and the maximum amplitude tends to vanish. Following a procedure used by Feng et al. [21] to introduce the so-called staggering transformation, i.e. \( j = (k_0 \omega_0) j / (x) \) and \( l = (k_0 \omega_0) l / (x) \) with \( k_0 = k / (x) \), to make a continuum approximation to slowly varying functions \( j \), \( l \) and \( k \) in \( x = j \), i.e. \( \phi(x + 1) = \phi(x) \pm \partial \phi / \partial x + (1/2) \partial^3 \phi / \partial x^2 + \cdots \), it follows from (3) that \( x_j - x_j - 1 \approx 1 - \phi_j^3/2 \) and \( y_j - y_j - 1 \approx (1/2)(-1) \partial \phi / \partial x \). Designating the small order of \( \phi \) by \( \epsilon (\ll 1) \), \( \partial / \partial x \) is also regarded as of order \( \epsilon \). Then it follows from (6) that

\[
\begin{align*}
\frac{1}{2} \frac{\partial^2 \phi^2}{\partial r^2} &= \frac{\partial^2 \lambda}{\partial x^2}, \\
\frac{1}{8} \frac{\partial^2}{\partial r^2} \left( \frac{\partial \phi}{\partial x} \right) &= \mu, \\
\frac{1}{12} \frac{\partial^2 \phi}{\partial r^2} &= -4\phi - \frac{\partial^2 \phi}{\partial x^2} - 16\kappa \phi^3 - \lambda \phi - \frac{1}{2} \frac{\partial \mu}{\partial x},
\end{align*}
\]
taken up to the order of $\varepsilon^3$.

Here the treatment of $\lambda$ is subtle and needs some comments. As far as no axial force is applied externally, $\lambda$ is brought about only by finite geometrical displacement and may be regarded as of order unity. As the axial distance $x_j - x_{j-1}$ between the centers of mass of neighboring units becomes shorter than unity by $\frac{x_j}{\varepsilon^2}/2$ because the unit is inextensible, the axial tension is determined by the sum of the inertia forces acting on each unit from one end of the structure located far away or at infinity. In fact, $\lambda$ is obtained by integrating (20a) with respect to $x$ as

$$
\lambda = -\int_{-\infty}^{\infty} (x - x') \frac{\partial^2}{\partial x'^2}\left(\frac{\phi^2}{2}\right)dx'.
$$

(21)

Because of integration, $\lambda\phi$ in (20c) might make a larger contribution than the order of $\varepsilon^3$. In fact, Fig. 14 shows the axial distributions of $k_j$ and $l_j$ in the case of the quasiperiodic oscillations with $j = 5000$ and $a = 0.6$ from $t = 193.00$ to $t = 193.24$ by step 0.04, where $l_j$ are depicted only at $t = 193.00$ and $t = 193.12$ because they appear to oscillate between the two profiles. Yet it will remain small in comparison with $16\mu_j^3$ if $\kappa$ is large. Incidentally if the constant axial force is applied externally as $\lambda = \lambda_0 + \cdots + \lambda_0$ being a constant, the term $\lambda_0\phi$ together with the first term on the right-hand side of (20c) will make a cutoff frequency change depending on the value of $\lambda_0$. If the axial force acts as compression and $\lambda_0$ is less than $-4$, the structure will become unstable to be buckled.

Keeping $\lambda\phi$ as it is, $\mu$ is eliminated in (20b) and (20c). Further applying the operator $1 - (1/4)\partial^2/\partial x^2$ to both sides of (20c) to neglect some terms of higher order, (20c) is reduced to the so-called regularized long-wave equation given by

$$
\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^4\phi}{\partial t^4\partial x^2} + 48\phi + 192\kappa\phi^3 + 12\lambda\phi = 0.
$$

(22)

If the last term $12\lambda\phi$ is negligible compared with $192\kappa\phi^3$, the equation is equivalent to the one derived in [21] with replacement $x$ by $x/2$. Thus the localized solutions are given by

$$
\phi = A \text{sech}(zx) \cos(\omega t),
$$

(23)

with $z = \sqrt{\kappa A^2/(2/3 + \kappa A^2)}$ and $\omega^2 = 48 + 72\kappa A^2$ in light of the breather solution of small amplitude obtained in [21] (see (30)). Note that no solutions exist for a soft spring ($\kappa < 0$). This solution suggests us to take the initial value given by (17). Thus as the amplitude $A$ becomes smaller, the frequency approaches $\sqrt{48}$, while
the spatial width $z^{-1}$ of the pulse becomes wider. The relation between $\omega$ and $A$ is drawn in Fig. 13 in thin chain lines for various values of $\kappa$. It is found that the primary angular frequencies $\omega_1$ are located close to these curves.

Here it should be noted that (23) represents time-harmonic oscillations of a single angular frequency. The other breather solution of large amplitude obtained in [21] indicates cnoidal oscillations but with a single frequency. Thus it is thought that the emergence of the quasiperiodic oscillations due to $\omega_2$ obtained numerically result from the nonlocal term $12z\phi$ due to the axial tension. In fact, as the value of $\kappa$ is decreased, no quasiperiodic oscillations appear. Such a nonlocal effect is given by (11) in the discrete case, and provides the essential difference of the articulated structure from the cases subjected to forces only due to the nearest neighbors.

To find whether or not the quasiperiodic oscillations exist irrespective of the number of units or existence of small oscillations in background, the calculations have been carried out for structures with a larger number of units. But it turns out that the Fourier spectrum in Fig. 9 change little if $N$ is increased up to $N = 96$. For this case with $\kappa = 5000$ and $z = 0.6$, Fig. 15 shows the comparison of the spatial decay of the maximum amplitude

Fig. 12. Fourier spectrum $|\tilde{\phi}_{32}(\phi)|$ of the oscillations in $\phi_{32}$ in (a) and Poincaré section in the phase plane $(\phi_{32}, \psi_{32})$ in (b) for the case with $\kappa = 100,000$ and $z = 0.6$. 
of $\phi_j$ in each unit for the cases with $N = 64$ and $N = 96$ indicated, respectively, by solid and blank circles. In passing, the calculations have been carried out for $1 \leq j \leq N/2$ only by exploiting the symmetry.

It is seen that the decay is exponential in space but there remain small oscillations even in the units near both ends of the structure, whose magnitude is comparable in both cases. It is also found that the magnitude of the small oscillations in background is of about 5% in comparison of the maximum at $j = N/2$. On the other hand, Fig. 9 shows the ratio of the Fourier spectrum at $\omega_2$ to the one at $\omega_1$ is about 1%. Analyzing the Fourier spectrum of the small oscillations of $\phi_j$ for the case with $N = 64$, it is revealed, as shown in Fig. 16, that most of the frequencies lie in the passing band below the cutoff frequency $\omega_0$, beyond which $|\phi_1^{(0)}|$ decays rapidly. Thus the small oscillations are propagated back and forth in the structure by reflections at both ends and are superimposed on the localized oscillations. Only in the case of a structure extending infinity, they would be radiated away to disappear. Thus it may be considered that the secondary frequency $\omega_2$ has nothing to do with the background oscillations.

Even if the emergence of the quasiperiodic oscillations is intrinsic, it is difficult to find a value of $\kappa$ at which the periodic solutions just bifurcate into quasiperiodic ones, because this judgment should rely on “by sight” of the Fourier spectrum only. Even in the case of the periodic solutions shown in Fig. 10, the Poincaré section has finite thickness. To make it thinner, the computation of high accuracy would be required. Further it is difficult to identify conditions for the delocalization to occur because they depend on the domain of computation in $t$ as well as initial conditions. Probably there would exist a “stochastic border” as in the Fermi–Pasta–Ulam β-model [22]. Then it is suggested that the border between quasi-periodic and chaotic solutions would not be sharp but rather wide and complicated.

Finally we discuss the order of errors in the numerical calculations. When the standard Runge–Kutta method of fourth-order is used, the relative error in the total energy does not remain within a certain level. Fig. 17 shows the relative errors of the total energy $E$ relative to the initial one $E_0$ in the cases of the quasiperiodic oscillations in Fig. 6 and the delocalized ones in Fig. 11. The error in Fig. 17(a) tends to increase linearly while oscillating with time, though each oscillation appears to be smeared out, and is less than $10^{-7}$ at

![Graph showing the relation between $\omega$ and $|\phi_{32}^{(i)}|$](image.png)
worst. On the other hand, the error in Fig. 17 (b) is worse by the order of $10^2$ than the former case but it remains within a certain level when the delocalization occurs. In any case, the linear growth of the errors is due to the nonsymplectic scheme.

Fig. 14. Axial distributions of $\lambda_j$ and $\mu_j$ in the case of the quasiperiodic oscillations with $\kappa = 5000$ and $\alpha = 0.6$ from $t = 193.00$ to $t = 193.24$ by step 0.04 in (a), and at $t = 193.00$ and $t = 193.12$ only in (b).

Fig. 15. Comparison of spatial decay of the maximum amplitude of $|\phi_j|$ in each unit ($1 \leq j \leq N$) in the interval $180 \leq t \leq 200$ for the cases with $N = 64$ and $N = 96$ indicated, respectively, by the solid and blank circles where $\kappa = 5000$, $A = \pi/180$ and $\alpha = 0.6$. 

On the other hand, the error in Fig. 17(b) is worse by the order of $10^2$ than the former case but it remains within a certain level when the delocalization occurs. In any case, the linear growth of the errors is due to the nonsymplectic scheme.
In parallel with the calculations presented, attempt has been made to solve (12) by using the symplectic partitioned Runge–Kutta method. Because it takes computation times about 50 times longer than the ones by the standard Runge–Kutta method, the computations are done for the three cases with \( \kappa = 1500, 5000 \) and \( \kappa = 10^{-9} \).

Fig. 16. Fourier spectrum \( |\phi_l^{(i)}| \) of the oscillations in \( \phi_t \) for the case with \( N = 64, \kappa = 5000 \) and \( \alpha = 0.6 \).

Fig. 17. Temporal variations of errors in the total energy \( E \) relative to the initial one \( E_0 \) by using the standard Runge–Kutta method: (a) and (b) show, respectively, the cases of the quasiperiodic oscillations and of the delocalized oscillations.
100,000 and $x = 0.6$. It is revealed that no visible differences are found in the figures for the profiles and spectra, though Poincaré section in the periodic case with $\kappa = 1500$ appears slightly thinner. But the errors in the total energy are improved surprisingly. Fig. 18 shows the temporal variations of errors in the symplectic integrations for the cases of the quasi-periodic oscillations and the delocalized ones. In Fig. 18(a), they remain always to be of order $10^{-10}$, though increasing very slowly. In Fig. 18(b), on the other hand, they are worse and of order $10^{-7}$ but the error is suppressed significantly when the delocalization occurs, just as in Fig. 17. As for the constrains (3), they are satisfied with absolute error of $10^{-16}$ at worst in the case of the symplectic integrations. When the standard Runge–Kutta method is applied, the errors are of order $10^{-4}$ in the worst case. In light of the computational times, however, it may be said that the standard Runge–Kutta method is useful and applicable enough to seek solutions up to $t = 200$.

### 5. Conclusions

The localized oscillations of the articulated structure subjected to the restoring moments at the couplers due to hard springs have been examined numerically with the free boundary conditions at both ends of the structure. It has been revealed that the stationary and spatially localized oscillations exist but the axial tensions at the junctions are not localized exponentially but algebraically. While the oscillations are periodic in the regime of small amplitude, it is expected that they bifurcate to become quasiperiodic in general. But it is open mathematically whether or not there exist exact quasiperiodic solutions corresponding to the ones obtained numerically, even in a periodic structure extending to infinity. Further it has been found that the localized
oscillations tend to be delocalized to exhibit chaotic behavior as the amplitude becomes larger. Such a behavior is not known in the ordinary ILMs in the lattice dynamics.

The results shown in this paper are restricted only to the symmetric profiles with respect to the $y$ axis by choosing the total number of the units to be even. But it should be mentioned again that qualitatively similar results are obtained in the case of the anti-symmetric profiles with odd number of units. In any case, the localized oscillations are stationary. Interestingly enough, however, it has been revealed that asymmetric initial disturbances begin to move in the structure. Up to the present, two types of such mobile localized oscillations are identified. One is propagated back and forth along the structure subjected to reflections at both ends, while the other is trapped by either one of the ends. Details will be published in a forthcoming paper.

Last but not least, it is significant that the localized oscillations are now shown to exist in the “bounded structure,” not in spatially periodic and infinitely extending structures. This implies that they would find many applications to engineering problems. In reality, then, there would exist small damping, more or less, and possible forcing. Nevertheless, they are expected to play a role of backbone in oscillations.

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References